International Journal of Innovation Sciences and Research
Vol.6, No, 02, pp. 987-990, February 2017
Available online at
http://www.ijisr.com

RESEARCH ARTICLE

YANG-MILLS EQUATIONS FOR A PULL BACK SYMPLECTIC YANG-MILLS FIELD

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Accepted 15th January, 2017; Published Online 28th February, 2017

ABSTRACT

We construct the pull back of a locally free A-module (E,π,Y) equipped with a symplectic A-form via a continuous map between two topological spaces, X and Y. We also derive, for a given symplectic Yang-Mills field (E,∇), the pull back symplectic Yang-Mills equations.

Key words: Symplectic Vector Sheaf, Symplectic A-Connection, A-Symplectomorphism.

INTRODUCTION

In [Jost, 2008], Jost defined the pull back bundle $f^*(E)$ from a bundle (E,π,Y) over Y and a continuous map $f : X \to Y$ between two topological spaces X and Y. In [2], Mallios developed the analog theory for vector sheaves. He established that for a given vector sheaf (E,π,Y), the inverse image $f^*(E)$ is also a vector sheaf. The purpose of this article is mainly to explore the pull back symplectic vector sheaf in detail. We suggest to consider (E,π,Y) a symplectic vector sheaf over a topological space Y, (A_Y,∂,Ω) a differential triad of Y, $\{(E, \nabla);\sigma\}$ a symplectic Yang-Mills field and $f : X \to Y$ a continuous map between the two topological spaces X and Y . After defining $(f^*(A_Y), f^*(\partial), f^*(\Omega))$ the differential triad of X, $\{(f^*(E), f^*(\nabla)); f^*(\sigma)\}$ the pull back symplectic Yang-Mills field and the pull back curvature $f^*(A_Y)$-tensor of $f^*(\nabla)$, we want to establish the Yang-Mills equations of $(f^*(E), f^*(\nabla))$. We show that the left action of the group sheaf of $f^*(A_Y)$-symplectomorphism of $f^*(E)$ on the affine space of $f^*(A_Y)$-connections on $f^*(E)$ provides the Yang-Mills field $(Sp(f^*(E), f^*(\nabla)_{Sp(f^*(E)})$ and permits to define the symplectic Yang-Mills equations. Special attention is given to the construction of $f^*(sric)$ the symplectic Ricci $f^*(A_Y)$-tensor on $f^*(E)$ via the pull back symplectic operator $f^*(sRic)$. Throughout this paper, $A_Y \equiv (A_Y,\tau,Y)$ is a sheaf of commutative, associative and unital ℂ-algebras over a topological space Y.

BASICS OF PULL BACK VECTOR SHEAF

In this section, we consider (Y,A_Y) an algebraized space relative to a topological space Y, (A_Y,∂,Ω) a differential triad of Y and a vector sheaf on Y.

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Definition 2.1 Let E ≡ (E, π, Y) be a vector sheaf on a topological space Y and $f : X \to Y$ a continuous map, the inverse image sheaf of E, denoted $f^*(E)$, is defined by $f^*(E) \equiv f^{-1}(E) = \{(x, z) \in X \times E : f(x) = \pi(z)\}$ where X is a topological space. (For more details, see for instance [Mallios, 1998], p79.). The inverse image sheaf of E, also named the pull back vector sheaf, is a subspace of X × E i.e $f^*(E) \subset X \times E$.

Definition 2.2 Given two vector sheaves (E,π,Y) and (F,ρ, Y), $\phi : E \to F$ a sheaf morphism and $f : X \to Y$ a continuous map, the map $f^*(\phi) : f^*(E) \to f^*(F)$ so that for every $f^*_V(t) \in f^*(E)(f^{-1}(V))$, $f^*(\phi)(f^*_V(t)) = f^*_V(\phi(t))$ is the pull back sheaf morphism of ϕ, where X and Y are topological spaces, t ∈ E(V) and V an open in Y . The sheaf morphism $f^*(\phi) : f^*(E) \to f^*(F)$ is fiber preserving, i.e for any x ∈ X, $f^*(\phi)(f^*(E))_x \subseteq (f^*(F))_x$.

Definition 2.3 Let E ≡ (E,π,Y) be a locally free A_Y -module on a topological space Y and (A_Y,∂, Ω) a differential triad of Y. The continuous map f: X→Y defines $(f^*(A_Y), f^*(\partial), f^*(\Omega))$ a differential triad on X. For $f^*_V(\alpha), f^*_V(\beta) \in f^*(A_Y)(f^{-1}(V))$, one gets $f^*(\partial)(f_V(\alpha).f_V(\beta)) = f^*(\partial)(f_V(\alpha)).f_V(\beta) + f_V(\alpha).f^*(\partial)(f_V(\beta))$ with $\alpha, \beta \in A_Y(V)$ and V an open subset of Y .

Definition 2.4 Given a continuous map $f : X \to Y$ between two topological spaces, E ≡ (E,π,Y) a locally free A_Y -module on Y, (A_Y,∂,Ω) a differential triad of Y and an A_Y-connection on E, the sheaf morphism $f^*(\nabla) : f^*(E) \to f^*(E)\otimes_{f^*(AY)}f^*(\Omega)$ so that for every $f_V(\alpha)\in f^*(A_Y)(f^{-1}(V))$ and $f_V(s)\in f^*(E)(f^{-1}(V))$, $f^*(\nabla)(f^*_V(\alpha).f^*_V(s)) = f^*_V(\alpha)f^*(\nabla)(f^*_V(s)) + f^*_V(s)\otimes f^*(\partial)(f^*_V(\alpha))$, is called an $f^*(A_Y)$-connection on $f^*(E)$ with $\alpha\in A_Y(V)$, $s \in E(V)$ and V an open subset of Y . Since $f^*(E)\otimes_{f^*(AY)}f^*(\Omega) = End_{f^*(AY)}f^*(E)$, we can write $f^*(\nabla) \in Hom(f^*(E), End_{f^*(AY)}f^*(E))$ and for any sections $f^*_V(t), f^*_V(s) \in f^*(E)(f^{-1}(V))$ we get $f^*(\nabla)(f^*_V(t))$.

$(f^*_V(s)) \in f^*(E)(f^{-1}(V))$ with t, s ∈ E(V) and V an open subset of Y . We notice that $f^*(\Omega)(\mathrm{End}_{f^*(AY)} f^*(E)) = \mathrm{Hom}(f^*(E), \mathrm{End}_{f^*(AY)} f^*(E))$. The set of $f^*(A_Y)$-connections on the inverse image sheaf of E is an affine space denoted by $\mathrm{Conn}_{f^*(AY)} f^*(E)$.

PULL BACK SYMPLECTIC VECTOR SHEAF

Proposition 3.1 Given f : X→Y be a continuous map between two topological spaces. Let E ≡ (E,π,Y) be a locally free A_Y -module on a topological space Y endowed with σ a symplectic A_Y -form. Then the inverse image sheaf of E is a locally free $f^*(A_Y)$-module on a topological space X equipped with the symplectic $f^*(A_Y)$ - form $f^*(\sigma)$.

Proof. Consider E ≡ (E,π,Y) a locally free A_Y -module on a topological space Y, i.e a symplectic vector sheaf (E,σ). From the definition (2.1), it appears that f*(E) is a vector sheaf on X. Now, we can show that $f^*(\sigma) : f^*(E)\oplus f^*(E)\rightarrow f^*(A_Y)$ is a symplectic $f^*(A_Y)$-form.

(i) For any $f^*_V(t), f^*_V(s) \in f^*(E)(f^{-1}(V))$, $f^*(\sigma)(f^*_V(s), f^*_V(t)) = f^*_V(\sigma(s, t))$
$= f^*_V(-\sigma(t, s))$
$= - f^*_V(\sigma(t, s))$
$= -f^*(\sigma)(f^*_V(t), f^*_V(s))$

with s, t ∈ E(V), V an open subset of Y.

(ii) For any $f^*_V(s) \in f^*(E)(f^{-1}(V))$, s ∈ E(V), V an open subset of Y, if $f^*(\sigma)(f^*_V(s),f^*_V(t)) = 0$, for all $f^*_V(t) \in f^*(E)(f^{-1}(V))$ i.e $f^*(\sigma)(f^*_V(s), f^*_V(t)) = 0$ for all t ∈ E(V), then t = 0 (we use the fact that σ is non degenerate). Thus, $f^*_V(t) = 0$ and we deduce that $f^*(\sigma)$ is a non-degenerate $f^*(A_Y)$ - form on f*(E). Hence, $f^*(\sigma)$ is a symplectic $f^*(A_Y)$-form on f*(E) and we conclude that $(f^*(E),f^*(\sigma))$ is a symplectic vector sheaf on the topological space X.

We recall that an A_Y-connection on a symplectic vector sheaf (E, σ) such that

$$\partial[\sigma(t, r)](s) = \sigma(\nabla (s)(t), r) + \sigma(t, \nabla (s)(r)) \quad (1)$$

is named a symplectic A_Y - connection, for any s, t, r ∈ E(V) and V an open subset of Y (see [5]).

Proposition 3.2 If ∇ is a symplectic A_Y-connection on a symplectic vector sheaf E ≡ (E,π,Y) ≡ (E, σ) and f : X →Y is a continuous map then $f^*(\nabla)$ is a symplectic $f^*(A_Y)$-connection on $(f^*(E),f^*(\sigma))$ where X and Y are two topological spaces.

Proof. Since ∇ is a symplectic A_Y-connection on E, equation (1) holds. The application of $f^*_V : A_Y \rightarrow f^*(A_Y)(f^{-1}(V))$ to (1) gives $f^*_V\partial[\sigma(t, r)](s) = f^*_V(\sigma(\nabla (s)(t), r)) + f^*_V(\sigma(t, \nabla (s)(r)))$, which can be rewritten either as

$$f^*(\partial)[f^*_V\sigma(t, r)](f^*_V(s)) = f^*(\sigma)(f^*(\nabla)(f^*_V(s))(f^*_V(t), f^*_V(r))+ f^*(\sigma)(f^*_V (t), f^*(\nabla)(f^*_V(s))(f^*_V(r)) \quad (2)$$

or as

$$f^*(\partial)[f^*_V\sigma(t,r)](f^*_V(s))= f^*(\partial)[f^*(\sigma)(f^*_V(t),f^*_V(r))](f^*_V(s)) \quad (3)$$

for any sections $f^*_V(t), f^*_V(r), f^*_V(s)$ of f*(E). It follows from the relations (2) and (3) that

$f^*(\partial)[f^*(\sigma)(f^*_V(t), f^*_V(r))](f^*_V(s))= f^*(\sigma)(f^*(\nabla)(f^*_V(s))(f^*_V(t), f^*_V(r))+ f^*(\sigma)(f^*_V (t), f^*(\nabla)(f^*_V(s))(f^*_V(r))$. Thus, $f^*(\nabla)$ is a symplectic $f^*(A_Y)$-connection on f*(E).

We remark that given {(E, ∇); σ)} a symplectic Yang-Mills field on Y, and f : X → Y a continuous map, $\{(f^*(E), f^*(\sigma)); f^*(\sigma)\}$ is a symplectic Yang-Mills field on the topological space X.

Proposition 3.3 Given two topological spaces X and Y, f: X→ Y a continuous map between them, E ≡ (E,σ) a symplectic vector sheaf over Y and $(f^*(E), f^*(\sigma))$ the inverse sheaf of (E, σ). If ψ is an A_Y-symplectomorphism of E then $f^*(\psi)$ is a $f^*(A_Y)$-symplectomorphism of f*(E).

Proof. By definition, ψ verifies the relation σ ∘ (ψ, ψ) = σ. Since ψ ∈ Sp(E) and f : X→ Y are continuous maps, one obtains $f^*(\psi) : f^*(E) \rightarrow f^* (E)$.

For any sections $f^*_V(t), f^*_V(r), f^*_V(s) \in f^*(E)(f^{-1}(V))$,

$f^*(\sigma) \circ (f^*(\psi),f^*(\psi))(f^*_V(t),f^*_V(s))$
$= f^*(\sigma) \circ (f^*(\psi)(f^*_V(t)), f^* (\psi)(f^*_V(s)))$,
$= f^*(\sigma)(f^*_V(\psi)(t), f^*_V(\psi)(s))$
$= f^*_V(\sigma(\psi(t), \psi(s)))$
$= f^*_V(\sigma(t, s))$
$= f^*(\sigma)(f^*_V(t), f^*_V(s))$.

Thus, $f^*(\psi)$ is a $f^*(A_Y)$ - symplectomorphism of f*(E).

We denoted by Sp(f*(E)) the group sheaf of $f^*(A_Y)$ - symplectomorphisms of f*(E). Using the action of the group sheaf of A_Y - symplectomorphisms of E on Conn_{AY} (E, σ) given by $\mathrm{Sp}(E) \times \mathrm{Conn}_{AY}(E, \sigma) \rightarrow \mathrm{Conn}_{AY}(E, \sigma)$, $(\varphi, \nabla) \rightarrow \nabla'= \varphi\circ\nabla\circ\varphi^{-1}$, the continuous map f: X →Y allows us to deduce the action of the group sheaf of $f^*(A_Y)$ -symplectomorphisms of f*(E) on $\mathrm{Conn}_{f^*(AY)}(f^*(E), f^*(\sigma))$, $\mathrm{Sp}(f^*(E)) \times \mathrm{Conn}_{f^*(AY)}(f^*(E),f^*(\sigma)) \rightarrow \mathrm{Conn}_{f^*(AY)}(f^*(E),f^*(\sigma))$,

$$f^*(\nabla') = f^*(\varphi)\circ f^*(\nabla)\circ f^*(\varphi^{-1}). \quad (4)$$

It is obvious that this action defines an equivalence relation on f*(E) by $f^*(\nabla) \sim f^*(\nabla')$ if and only if there exits

$f^*(\varphi) \in \mathrm{Sp}(f^*(E))$ such that $f^*(\nabla') = f^*(\varphi)\circ f^*(\nabla)\circ f^*(\varphi^{-1})$

for any $f^*(\nabla), f^*(\nabla') \in \mathrm{Conn}_{f^*(AY)} (f^*(E),f^*(\sigma))$.

The quotient $\mathrm{Conn}_{f^*(AY)}(f^*(E),f^*(\sigma))/\mathrm{Sp}(f^*(E))$ is called the moduli space of the symplectic $f^*(A_Y)$-connections on f*(E) and the equivalence class of $f^*(\nabla)$ or its orbit is the following set:

$$[f^*(\nabla)] = \{ f^*(\nabla') = f^*(\varphi)\circ f^*(\nabla)\circ f^*(\varphi^{-1}), f^*(\varphi) \in \mathrm{Sp}(f^*(E))\} \quad (5)$$

We remark that a $f^*(A_Y)$ - connection on f*(E) induces the following $f^*(A_Y)$ - connection on Sp(f*(E)), $f^*(\nabla)_{Sp(f^*(E))}(f^*(\varphi)) = f^*(\nabla) \circ f^*(\varphi) - (f^*(\varphi) \otimes 1_{f^*(\Omega)}) \circ f^*(\nabla)$ where $f^*(\varphi) \in \mathrm{Sp}(f^*(E))$, φ ∈ Sp(E) and f : X →Y a continuous map. The symplectic Yang-Mills field $(f^*(E),f^*(\nabla))$ provides the Yang-Mills field $(\mathrm{Sp}(f^*(E)), f^*(\nabla)_{Sp(f^*(E))})$. Recall that the first prolongation of a symplectic A_Y-connection ∇on a symplectic vector sheaf (E,σ) is given by $\nabla^1 : E\otimes_{AY}\Omega \rightarrow E\otimes_{AY}\Omega^2$. From f: X → Y a continuous map and ∇^1, one gets

$f^*\nabla^1$): $f^*(E)\otimes_{f^*AY}f^*(\Omega) \rightarrow f^*(E)\otimes_{f^*AY}f^*(\Omega^2)$ the first prolongation of $f^*(\nabla)$.

Definition 3.4 Let $f : X \rightarrow Y$ be a continuous map two topological spaces, $E \equiv (E,\pi,Y)$ a locally free A_Y - module on Y endowed with a symplectic A_Y - form σ and ∇an A_Y - connection on (E,σ), the curvature of the symplectic $f^*(A_Y)$ - connection is defined by $R(f^*(\nabla)) = f^*(\nabla^1) \circ f^*(\nabla)$.

The pull back preserves the curvature in sense of $R(f^*(\nabla)) = f^*(R(\nabla))$ (see [3], p.235).

Definition 3.5 Given $f : X\rightarrow Y$ a continuous map, (E,π,Y) a symplectic vector sheaf over Y and an A_Y - connection on E, the curvature $f^*(A_Y)$ - tensor of $f^*(\nabla)$ is defined by

$$R_{|f-1(V)}(f^*_V(s), f^*_V(t))\, f^*_V(r) = R(f^*_V(s), f^*_V(t))\, f^*_V(r)$$
$$= f^*(\nabla)(f^*_V(s))\, f^*(\nabla)(f^*_V(t)) - f^*(\nabla)(f^*_V(t))\, f^*(\nabla)(f^*_V(s)) - f^*(\nabla)([f^*_V(s), f^*_V(t)])\, f^*_V(t)\,(r), \quad (6)$$

for any $f^*_V(s), f^*_V(t), f^*_V(r) \in f^*(E)(f^{-1}(V))$, $s, t, r \in E(V)$ and V an open subset of Y.

The last relation (6) can be written as

$$R_{|f-1(V)}(f^*_V(s), f^*_V(t))\, f^*_V(r)$$
$$=(f^*_V(\nabla)(s)f^*_V(\nabla)(t)- f^*_V(\nabla)(t)f^*_V(\nabla)(s)-[f^*_V(\nabla)(s),f^*_V(\nabla)(t)])$$
$$f^*_V(r), \quad (7)$$

for any $f^*_V(s), f^*_V(t), f^*_V(r) \in f^*(E)(f^{-1}(V))$, $s, t, r \in E(V)$ and V an open subset of Y. The curvature operator associated with the symplectic Yang-Mills field $(f^*(E),f^*(\nabla))$ is defined by $R_{|f-1(V)}(.\,, f^*_V(s))f^*_V(t) = R(f^*_V(s))\, f^*_V(t) \in End_{f^*(AY)}$

$$(f^*(E)(f^{-1}(V))), \quad (8)$$

for any $f^*_V(s), f^*_V(t), f^*_V(r) \in f^*(E)(f^{-1}(V))$, $s, t, r \in E(V)$ and V an open subset of Y.

Definition 3.6 Let ric be the Ricci curvature A_Y - tensor of an A_Y -connection ∇ on E and let $f : X \rightarrow Y$ be a continuous map between topological spaces.The Ricci curvature $f^*(A_Y)$-tensor on $f^*(E)$, denoted by $f^*(ric)$, is defined by $f^*(ric)_{|f-1(V)}(f^*_V(s),$

$$f^*_V(t)) = tr(f^*_V(r) \rightarrow R(f^*_V(r), f^*_V(s))\, f^*_V(t) \quad (9)$$

for any $f^*_V(s), f^*_V(t), f^*_V(r) \in f^*(E)(f^{-1}(V))$, $s, t, r \in E(V)$ and V an open subset of Y.

Definition 3.7 Let sR be the symplectic curvature A_Y -tensor associated with the curvature of a symplectic A_Y-connection on a symplectic vector sheaf $E \equiv (E,\pi,Y) \equiv (E,\sigma)$ and let $f : X\rightarrow Y$ be a continuous map between topological spaces,

$f^*(sR)$ defined by

$$f^*(sR)_{|f-1(V)}(f^*_V(s), f^*_V(t), f^*_V(r), f^*_V(l))$$
$$= f^*(\sigma)(R_{|f-1(V)}(f^*_V(s), f^*_V(t))\, f^*_V(r), f^*_V(l)),$$

is the symplectic curvature $f^*(A_Y)$ - tensor relative to the $f^*(A_Y)$ - connection $f^*(\nabla)$, for any $f^*_V(s), f^*_V(t), f^*_V(r), f^*_V(l) \in f^*(E)(f^{-1}(V))$, $s, t, r, l \in E(V)$ and V an open subset of Y .

Referring toa local gauge $e^V = \{V;e_1, e_2, \ldots, e_{2n}\}$ of a symplectic vector sheaf $E \equiv (E,\pi,Y) \equiv (E, \sigma)$ of rank 2n where V is an open subset of Y , the continuous $f : X \rightarrow Y$ defines $f^*(e^V) = \{f^{-1}(V); f^*(e_1), f^*(e_2), .. , f^*(e_{2n})\}$ a local gauge of the inverse image sheaf $f^*(E)$.

Proposition 3.8 Let $E \equiv (E,\pi,Y)$ be a vector sheaf over a topological space Y and $f : X \rightarrow Y$ be a continuous map. If $J : E \rightarrow E$ is an A_Y - complex structure on E then the pull back of J is a $f^*(A_Y)$ - complex structure on $f^*(E)$.

Proof. The pull back of J is the sheaf morphism

$f^*(J) : f^*(E) \rightarrow f^*(E)$ so that

$$\begin{aligned}(f^*(J))^2(f^*_V(s)) &= f^*(J)(f^*(J)(f^*_V(s)) \\ &= f^*(J)(f^*_V J(s)) \\ &= f^*_V(J(J(s)) \\ &= f^*_V(J^2(s)) \\ &= f^*_V(-id_{E(V)}(s)) \\ &= - f^*_V(s), \end{aligned} \quad (10)$$

for any $f^*_V(s) \in f^*(E)(f^{-1}(V))$, $s \in E(V)$ and V an open subset of Y. Thus $f^*(J)$ is a $f^*(A_Y)$ - complex structure on $f^*(E)$.

Definition 3.9 Let sRic be the symplectic Ricci operator of $E \equiv (E,\sigma)$ and $f: X \rightarrow Y$ a continuous map between two topological spaces, $f^*(sRic)$, is defined by $f^*(sRic)|_{f-1(V)}(f^*_V(s)) = \sum(f^*_V(e_i),$

$$f^*(J)f^*_V(e_i))\, f^*_V(s),\ i=1,2,\ldots,n, \quad (11)$$

it is the symplectic Ricci operator of $f^*(E)$, for any $f^*_V(s) \in f^*(E)(f^{-1}(V))$, $s \in E(V)$, V an open subset of Y and $f^*(J)$ the pull back of the complex structure J of E.

We also define the symplectic Ricci $f^*(A_Y)$-tensor on $f^*(E)$ as follows :

$$f^*(sric)|_{f^{-1}(V)}(f^*_V(s), f^*_V(t)) = f^*(\sigma)(sRic(f^*_V(s), f^*_V(t)) \quad (12)$$

for any $f^*_V(s), f^*_V(t) \in f^*(E)(f^{-1}(V))$, $s, t \in E(V)$ and V an open subset of Y.

PULL BACK OF SYMPLECTIC LAPLACE - BELTRAMI OPERATOR

Consider (Y,A_Y) a ℂ-algebraized space, (A_Y,∂,Ω) a differential triad of Y, a continuous map $f : X \rightarrow Y$ defines $(f^*(A_Y),f^*(\partial),f^*(\Omega))$ a differential triad of X from which we derive the m^{th} - prolongation of $f^*(\partial)$, denoted by $f^*(d^m)$. Since the m^{th} - prolongation of $d^m : \Omega^m \rightarrow \Omega^{m+1}$ of $\partial : AY \equiv \Omega^0 \rightarrow \Omega^1$ is defined by

$$d^m(s \wedge t) = d^{p+q}(s \wedge t) = d^p(s) \wedge t + (-1)^p s \wedge d^q(t) \quad (13)$$

for every $s \in \Omega^p(V)$, $t \in \Omega^q(V)$, $p, q \in$ IN and V open in Y (see for instance [4]), one obtains the pull back of d^m,

$d^m : \Omega^m \rightarrow \Omega^{m+1}$ so that

$$f^*(d^m)(f^*_V(s) \wedge f^*_V(t)) = f^*(d^{p+q})(f^*_V(s) \wedge f^*_V(t))$$
$$= f^*(d^p)(f^*_V(s)) \wedge f^*_V(t) + (-1)^p f^*_V(s) \wedge f^*(d^q)(f^*_V(s))$$
$$= f^*_V(d^p(s)) \wedge f^*_V(t) + (-1)^p f^*_V(s) \wedge f^*_V(d^q(t)) \quad (14)$$

for any $f^*_V(s) \in f^*(\Omega^p)(f^{-1}(V))$, $f^*_V(t) \in f^*(\Omega^q)(f^{-1}(V))$ and V open in Y.

We recall that for a given vector sheaf E over Y, the m^{th} - prolongation of the A_Y - connection $\nabla \equiv \nabla^0 : \Omega^0(E) \rightarrow \Omega^1(E)$, $\nabla^m : \Omega^m(E) \rightarrow \Omega^{m+1}(E)$ is defined by

$$\nabla^m(s \otimes t) = s \otimes d^m(t) + (-1)^m t \otimes \nabla(s) \quad (15)$$

for any $s \in E(V)$, $t \in \Omega^m(V)$ and V open in Y (see [4]).

Definition 4.1 Let (E, ∇) be a Yang-Mills field over a topological space Y, $\nabla^m : \Omega^m(E) \rightarrow \Omega^{m+1}(E)$ the m^{th} - prolongation of the A_Y - connection ∇ and $f : X \rightarrow Y$ a continuous map between two topological spaces, the pull back of ∇, denoted $f^*(\nabla^m)$, is defined from by

$$f^*(\nabla^m) : f^*(\Omega^m)(f^*(E)) \rightarrow f^*(\Omega^{m+1})(f^*(E)),$$
$$f^*(\nabla^m)(f^*_V(s) \otimes f^*_V(t)) = f^*_V(s) \otimes f^*_V(d^m)(f^*_V(t)) + (-1)^m f^*_V(t) \otimes f^*(\nabla)(f^*_V(s)) \quad (16)$$

for any $f^*_V(s) \in f^*(E)(f^{-1}(V))$, $f^*_V(t) \in f^*(\Omega^m)(f^{-1}(V))$, and V open in Y .

We can rewrite (16) as follows:

$$f^*(\nabla^m)(f^*_V(s) \otimes f^*_V(t)) = f^*_V(s) \otimes f^*_V(d^m(t)) + (-1)^m f^*_V(t) \otimes f^*(\nabla(s)) \quad (17)$$

for any $f^*_V(s) \in f^*(E)(f^{-1}(V))$, $f^*_V(t) \in f^*(\Omega^m)(f^{-1}(V))$, and V open in Y .

We recall that on (E,σ) a symplectic vector sheaf over Y, the dual differential operator of ∇, $\delta^{m+1} : \Omega^{m+1}(E) \rightarrow \Omega^m(E)$ is such that

$$\sigma\nabla^m(s), t) = \sigma(s, \delta^{m+1}(t)), \quad (18)$$

for any $s \in \Omega^m(E(V))$, $t \in \Omega^{m+1}(E(V))$, with V open in Y (see [5]).

Definition 4.2 Let $\{(E, \nabla),\sigma\}$ be a symplectic Yang-Mills field over a topological space Y, δ^{m+1} the dual differential operator of $\nabla$and $f : X \rightarrow Y$ a continuous map between two topological spaces, $f^*(\delta^{m+1}) : f^*(\Omega^{m+1}(E)) \rightarrow f^*(\Omega^m(E))$ is such that

$$f^*(\sigma)(f^*(\nabla^m)(f^*_V(s), f^*_V(t)) = f^*(\sigma)(f^*_V(s), f^*(\delta^{m+1})(f^*_V(t))) \quad (19)$$

is the pull back of δ^{m+1}, for every $f^*_V(s) \in f^*(\Omega^m(E))(f^{-1}(V))$, $f^*_V(t) \in f^*(\Omega^{m+1}(E))(f^{-1}(V))$ and V open in Y .

(19) can be expressed as follows

$$f^*(\sigma)(f^*(\nabla^m)(f^*_V(s), f^*_V(t)) = f^*(\sigma)(f^*_V(s), f^*_V(\delta^{m+1}(t)), \quad (20)$$

$$f^*_V(\sigma(\nabla^m(s), t)) = f^*_V(\sigma(s, \delta^{m+1}(t)), \quad (21)$$

for any $f^*_V(s) \in f^*(\Omega^m(E))(f^{-1}(V))$, $f^*_V(t) \in f^*(\Omega^{m+1}(E))$ and V open in Y.

Hence,

$$f^*(\sigma)(f^*(\nabla^m)(f^*_V(s), f^*_V(t)) = f^*_V(\sigma(s, \delta^{m+1}(t)).$$

From the symplectic Laplace-Beltrami operator correspon-ding to a symplectic A_Y - connection ∇on a symplectic vector sheaf (E, σ),

$$\Delta^m = \delta^{m+1} \circ \delta^m \quad (22)$$

(see[4]) and a continuous map $f : X \rightarrow Y$ between topologi-cal spaces, one gets the pull back of Δ^m ,

$$f^*(\Delta^m) = f^*(\delta^{m+1}) \circ f^*(\nabla^m) + f^*(\nabla^{m-1}) \circ f^*(\delta^m) \quad (23)$$

which is the symplectic Laplace-Beltrami operator corresponding to the $f^*(A_Y)$-connection $f^*(\nabla)$ on $(f^*(E),f^*(\sigma))$.

It follows that the Laplace-Beltrami operator corresponding to the $f^*(A_Y)$-connection $f^*(\nabla)_{Sp(f^*(E))}$ on the Yang-mills field $(Sp(f^*(E)), f^*(\nabla)_{Sp(f^*(E))})$ is given by

$$f^*(\Delta^m_{Sp(f^*(E))}) = f^*(\delta^{m+1}_{Sp(f^*(E))}) \circ f^*(\nabla^m_{Sp(f^*(E))}) + f^*(\nabla^{m-1}_{Sp(f^*(E))}) \circ f^*(\delta^m_{Sp(f^*(E))}). \quad (25)$$

In particular for m = 2, one gets

$$f^*(\Delta^2_{Sp(f^*(E))}) = f^*(\delta^3_{Sp(f^*(E))}) \circ f^*(\nabla^2_{Sp(f^*(E))}) + f^*(\nabla_{Sp(f^*(E))}) \circ f^*(\delta^2_{Sp(f^*(E))}) \quad (26)$$

which is the pull back for

$\Delta^2_{Sp(E)} = \delta^3_{Sp(E)} \circ \nabla^2_{Sp(E)} + \nabla_{Sp(E)} \circ \delta^2_{Sp(E)}$ developed in [5].

Hence, the Yang-Mills equations of $(f^*(E),f^*(\nabla))$ are

$$f^*(\Delta^2_{Sp(f^*(E))})(R(f^*(\nabla))) = 0 \quad (27)$$

and

$$f^*(\delta^2_{Sp(f^*(E))})(R(f^*(\nabla))) = 0 \quad (28)$$

where $R(f^*(\nabla)) = f^*(R(\nabla))$ is the pull back of $R(\nabla)$.

Conclusion

In this paper, more details and results about the pull back symplectic vector sheaf and the pull back symplectic Yang-Mills field are given. We mainly apply the pull back symplectic Laplace-Beltrami operator to define the symplectic Yang-Mills equations on the Yang-Mills field $(f^*(E),f^*(\nabla))$.

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