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RESEARCH ARTICLE

THE QUADRATIC DIFFERENTIAL TRIADS VIA SHEAVES OF SETS AND CATEGORIES

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ABSTRACT

The determination of quadratic forms throughout sheaves $\mathcal{A}$ and Ω, and functorial morphisms, play an important role in the construction of some categories of quadratic differential triads over *X*, id_X, $id_{X\times id_X}$, ... In this paper, we generalize these notions so that we obtain categories of quadratic differential triads over the categories $Open_X$ and *TOP*, respectively. We also treat the notions of quadratic integral triads. The cohomology, homology and resolution notions are introduced with some applications in Electromagnetism.

Key words: Quadratic differential triads, morphism of quadratic differential triads, categories of quadratic differential triads, quadratic integral triads and category of quadratic integral triads

INTRODUCTION

The sheaves and presheaves are treated over fixed topological spaces and their extensions are treated over fixed categories of topological spaces. We associate to a sheaf or a presheaf a quadratic form associated to a bilinear form or generally to an hermitian form. We use the complete presheaf to identify it, with the help of isomorphism, to a sheaf. The association of quadratic forms $q_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{A}$ and $q_{\Omega} : \Omega \rightarrow \mathcal{A}$ of sheaves $\mathcal{A}$ and Ω, respectively , transforms a differential triad $dT_X = (\mathcal{A}_X , d_X , \Omega_X)$ over $(X, \mathcal{A}_X)$ to a differential triad $qdT_X = ((\mathcal{A}_X , q_{\mathcal{A}_X}) , d_X , (\Omega_X , q_{\Omega_X}))$ called a quadratic differential triad over $(X, (\mathcal{A}_X , q_{\mathcal{A}_X}))$. The composition law and the identity morphism play an important role in the construction of categories QDT_X , QDT_{Id_X}, $QDT_{Hom(X,Y)}$, QDT_{Open_X} and QDT_{TOP} of quadratic differential triads. From the reciprocal relation of the differential map d , said $\int$, we define the quadratic integral triad and consequently the category of integral triad.

Definitions 1.1 Let *X* be a fixed topological space. A *sheaf of sets* over *X*, is defined as the triplet

$$(S, s, X) \qquad \text{[1.1]}$$

such that, $s: S \rightarrow X$ is a surjective (local) homeomorphism as treated in [11].

A *presheaf P of* **sets** on *X* is an assignment (correspondence) that associates a set *P(U)* to every open subset *U* of *X*, where the following conditions are satisfied [11]:

- For any open sets *U,V* of *X*, with $V \subseteq U$, there exists a restriction map $\delta_V^U : P(U) \rightarrow P(V)$
- For every open set *U* of *X*, $\delta_U^U = id_{F(U)}.$
- For any open sets *U, V, W* in *X*, with $W \subseteq V \subseteq U$, $\delta_W^U = \delta_W^V \circ \delta_V^U$

If *S* is a sheaf on a topological space *X*, then $S(U) \equiv \Gamma(U; S)$ stands for the set of local sections of S on *U* and we set $(S(U); \delta_V^U) \equiv (\Gamma(U; S); \delta_V^U)$ to be the *presheaf of sections of S*, where δ_V^U is the restriction map.

Let $\Gamma_S \equiv \Gamma(S) = (S(U); \sigma_V^U)$ be a presheaf of sets on a topological space *X*. Then, Γ_S is a *complete presheaf* if the following conditions are satisfied [11] and [25]:

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- If U is an open subset of X and $(Ui)_{i\epsilon I}$ is an open covering of U; let $s_1, s_2 \epsilon S(U)$ such that $\sigma^U_{Ui}(s_1) = \sigma^U_{Ui}(s_2)$, for every $i \epsilon I$, then $s_1 = s_2$ (the converse is certainly true).
- Let U and $(Ui)_{i\epsilon I}$ be as defined in (1); moreover let $(s_i) \epsilon \prod_i S(Ui)$ so that, for any $U_{ij} \equiv U_i \cap U_j \neq \emptyset$; in $(Ui)_{i\epsilon I}$, one has: $\sigma^{Ui}_{Uij}(s_i) = \sigma^{Uj}_{Uij}(s_j)$

Then, there exists an element $s \epsilon S(U)$ such that $\sigma^U_{Ui}(s) = s_i$, for all $i \epsilon I$.

Remarks 1.2

For any $x \epsilon X$, one has

$$s^{-1}(x) := S_x \qquad \text{[1.2]}$$

S_x is a fiber of S over x or a stalk of S at x

We set

$$S = \cup_{x\epsilon X} S_x \equiv \textstyle\sum_{x\epsilon X} S_x \qquad \text{[1.3]}$$

If S_x is a Group (Ring, vector space, module, algebra,…), then (S, s, X) is a sheaf of Groups (Rings, vector spaces , modules, algebras,…) over X and for each case we have :

$$S_x = \underset{x\epsilon U}{\underleftarrow{lim}} S(U) \equiv \underset{x\epsilon X}{\underleftarrow{lim}} S(U), \text{ with } x \epsilon U \subseteq X,$$

where $\underset{x\epsilon X}{\underleftarrow{lim}}$ represents the *inductive limit*.

Notation 1.3 We denote a sheaf of sets over X and the associate presheaf of sections of S, respectively by

$$S_X = (S, s, X) \text{ and } \Gamma_S \equiv \Gamma(S) = (S(U); \delta^U_V) \qquad \text{[1.4]}$$

Let $(\mathcal{A}, a, X)$ be a sheaf of $\mathbb{C}_X$–algebras (or in other words, a $\mathbb{C}_X$–algebra sheaf), which is preferably unital and commutative, with $\mathbb{C}_X$ the sheaf of complex numbers over X. We design by $\mathcal{A}^+{}_X$ and $\mathcal{A}^-{}_X$ the sub sheaves of $\mathcal{A}_X$ formed by positive elements and negative elements, respectively. Thus:

$$\mathcal{A}^+{}_X \cap \mathcal{A}^-{}_X = \{0\}_X, \ \mathcal{A}^+{}_X \cup \mathcal{A}^-{}_X = \mathcal{A}_X \qquad \text{[1.5]}$$

Definitions 1.4 The Sheaf $S_X = (S, s, X)$ is a *vector sheaf*, if, for any open $U \subseteq X$, we have

$$S_U \equiv S(U) \cong \mathcal{A}^n(U) \equiv \mathcal{A}^n{}_U \qquad \text{[1.6]}$$

The presheaf $\Gamma_{\mathcal{A}} = (\mathcal{A}(U); a^U_V)$ is a *vector presheaf*, if , for any open $U \subseteq X$, we have

$$\Gamma_S = (S(U); \delta^U_V) \cong (\mathcal{A}(U); a^U_V)^n = \Gamma_{\mathcal{A}}{}^n \qquad \text{[1.7]}$$

Definitions 1.5 Let (S, s, X) and (Σ, σ, X) be two sheaves of sets over X. A morphism φ of sheaves (or simply, a sheaf morphism) from (S, s, X) to (Σ, σ, X) is a continuous map:

$$\varphi : S \rightarrow \Sigma$$

such that

$$\varphi(S) \subseteq \Sigma, \quad \sigma \circ \varphi = s. \qquad \text{[1.8]}$$

Let $\Gamma_S \equiv (S(U); \delta^U_V)$ and $\Gamma_\Sigma \equiv (\Sigma(U); \sigma^U_V)$ be two presheaves of sections of S and Σ, respectively.

A morphism of presheaves (or simply, a presheaf morphism) from $\Gamma_S \equiv (S(U); \delta^U_V)$ to $\Gamma_\Sigma \equiv (\Sigma(U); \sigma^U_V)$ is defined as a continuous map

$$\Gamma_\varphi \equiv \Gamma(\varphi) : \Gamma_S \equiv \Gamma(S) \rightarrow \Gamma(\Sigma\Sigma) \equiv \Gamma_\Sigma \qquad \text{[1.9]}$$

such that

$$\varphi_U(S(U)) \subseteq \Sigma(U) \text{ and } \varphi_V \, o \, \delta^U_V = \sigma^U_V \, o \, \varphi_U \qquad \text{[1.10]}$$

We design by Sh_X, PSh_X and $CoPSh_X$ the categories of sheaves, presheaves and complete presheaves, respectively. The functor $\Gamma: Sh_X \rightarrow CoPSh_X \subseteq PSh_X$, $S \rightarrow \Gamma_S$ is an isomorphism called the *section functor* and by isomorphism between Sh_X and $CoPSh_X$, we define a functor $S: CoPSh_X \rightarrow Sh_X$, *with* $S \circ \Gamma \equiv S\Gamma = id_{Sh_X}$ called the *she afification functor* and satisfying the relation :

$S\Gamma(\Sigma) \equiv S(\Gamma(\Sigma) = \Sigma$, *for any* $\Sigma \in Sh_X$.

Definition 1.6 Let *X* be a fixed topological space. A *sheaf* of categories over *X*, is defined as a triplet *(C, c, X)* such that, $c : C \rightarrow X$ is a surjective (local) homeomorphism ([6], [9], [12]). A *presheaf P* of categories on *X* is an assignment (correspondence) that associates a category of sets *P(U)* to every open subset *U* of *X*, where the following conditions are satisfied:

- For any open sets *U, V* of *X*, with $V \subseteq U$, there exists a restriction functor $\delta_V^U : P(U) \rightarrow P(V)$, with $\delta_V^U: Ob(P(U) \rightarrow Ob(P(V)$ and $\delta_V^U: Mor(P(U) \rightarrow Mor(P(V)$
- For every open set *U* of *X*, $\delta_U^U = id_{P(U)}$.
- For any open sets *U, V, W* in *X*, with $W \subseteq V \subseteq U$, $\delta_W^U = \delta_W^V \circ \delta_V^U$

If *S* is a sheaf of categories on a topological space *X*, then:

$S(U) \equiv \Gamma(U; S)$,

with $Ob(S(U)) \equiv Ob(\Gamma(U; S))$ and $Mor(S(U)) \equiv Mor(\Gamma(U; S))$, stands for *the category of sets of local sections of S* on U, and we set $(S(U); \delta_V^U) \equiv (\Gamma(U; S); \delta_V^U)$, to be a category presheaf of sections of *S*, with $Ob((S(U); \delta_V^U)) \equiv Ob((\Gamma(U; S); \delta_V^U)$ and $Mor((S(U); \delta_V^U)) \equiv Mor((\Gamma(U; S); \delta_V^U))$, where δ_V^U is the restriction functor.

Let $\Gamma_P \equiv \Gamma(P) = (P(U); \sigma_V^U)$ be a presheaf (of categories) on a topological space *X*. Then, *P* is a *complete presheaf* if the following conditions are satisfied:

(1) If *U* is an open subset of *X* and $(Ui)_{i \in I}$ is an open covering of *U*; let $s_1, s_2 \in P(U)$ such that $\sigma_{Ui}^U i(s_1) = \sigma_{Ui}^U(s_2)$, for every $i \in I$, then $s_1 = s_2$ (the converse is certainly true);

(2) Let *U* and $(Ui)_{i \in I}$ be as in (1); moreover let $(s_i) \in \prod_i P(U_i)$ such that, for any $U_{ij} \equiv U_i \cap U_j \neq \emptyset$; in $(Ui)_{i \in I}$, we have: $\sigma_{Uij}^{Ui}(s_i) = \sigma_{Uij}^{Uj}(s_j)$

Then, there exists an element $s \in F(U)$ such that $\sigma_{Ui}^U(s) = s_i$, for all $i \in I$.

Remark 1.7 For any $x \in X$, we have:

$$c^{-1}(x) := C_x \quad \text{[1.11]}$$

and

$$C := \bigcup_{x \in X} C_x \equiv \sum_{x \in X} C_x \quad \text{[1.12]}$$

Also, if C_x is a category of sets (topological spaces, commutative groups, modules, algebras…), then *(C, c, X)* is a sheaf of categories of sets (topological spaces, commutative groups, modules, algebras…). Thus, as C_x is a category, then we form the following sets said $Ob(C_x)$ and $Mor(C_x)$ such that :

$$Ob(C) = \sum_{x \in X}(Ob(C_x), \; Mor(C) = \sum_{x \in X}(Mor(C_x)) \quad \text{[1.13]}$$

From presheaves theory, we set : $Ob(\Gamma_P) = Ob((P(U); \sigma_V^U))$, $Mor(\Gamma_P) = Mor((P(U); \sigma_V^U))$

Definition 1.8 Let *(C, c, X)* and *(D, d, X)* be two sheaves of categories over *X*. A morphism of sheaves (or simply, a sheaf morphism) from *(C, c, X)* to *(D, d, X)* is a continuous functor $: C \rightarrow D$ such that $F(C) \subseteq D$

with *:*

$$d \circ F = c \quad \text{[1.14]}$$

Let $\Gamma_C \equiv \Gamma(C) = (C(U); c_V^U)$ and $\Gamma_D \equiv \Gamma(D) = (D(U); d_V^U)$ be too presheaves of categories on *X*. A morphism of presheaves (or simply, a presheaf morphism) from $\Gamma_C \equiv \Gamma(C) = (C(U); c_V^U)$ to $\Gamma_D \equiv \Gamma(D) = (D(U); d_V^U)$ is a continuous functor $\Gamma_F : \Gamma_C \rightarrow \Gamma_D$ such that :

$$d_V^U \circ F_U = F_V \circ c_V^U \quad \text{[1.15]}$$

where F_U: $C(U) \rightarrow D(U)$ and F_V: $C(V) \rightarrow D(V)$ are morphisms in the category of sets.

We observe in [1.14] and [1.15] that there are compositions of a morphism D with a functor F to obtain a morphism c; i.e., that represents the composition of mathematical objects of different natures. This remains an open problem which needs a special attention. To address this question, we suggest to replace the topological space X by the category of opens of X, denoted $Open_X$ or in general by the category, denoted *TOP*, of topological spaces.

In the following, the correspondence from complete presheaves to sheaves and vice versa is possible through the sheafification functor $S\colon CoPSh_X \rightarrow Sh_X$ or the section functor $\Gamma\colon Sh_X \rightarrow CoPSh_X \subseteq PSh_X$, with $S \, o \, \Gamma \equiv S\Gamma = id_{Sh_X}$. With these regards, all sheaves are considered to be generated by complete presheaves.

II. Quadratic Differential triads

Some notions of this section are treated classically in [1] [3] [5] [7] [8] [10] [16-18] [26] [27] and in Abstract Algebra in [12-13], [25].

Definitions 2.1 Let E_X be a free $\mathcal{A}_X$-module. A map $b_X\colon E_X \oplus E_X \rightarrow \mathcal{A}_X$ is an $\mathcal{A}_X$-*bilinear form*, if there exist, for any $s, t \in E_U \equiv E(U)$, with $U \subseteq X$ open, two $\mathcal{A}_U$-linear forms $b_{s,U}\colon E_U \rightarrow \mathcal{A}_U$, $t \rightarrow b_{s,U}(t)$ and $b_{t,U}\colon E_U \rightarrow \mathcal{A}_U$, $t \rightarrow b_{t,U}(s)$ such that

$$b_U(s,t) = b_{s,U}(t) = b_{t,U}(s), \qquad \text{.........................[2.1]}$$

where $b_U\colon E_U \oplus E_U \rightarrow \mathcal{A}_U$ is an $\mathcal{A}_U$- bilinear form which satisfies the following

$$a_V^U \, o \, b_U = b_V \, o \, (e_V^U \times e_V^U), \text{ with } V \subseteq U \text{ open.} \qquad \text{...........................[2.2]}$$

Definition 2.2 Let E_X be a free $\mathcal{A}_X$-module (or a vector sheaf). A sheaf morphism $q_{E_X}\colon E_X \rightarrow \mathcal{A}_X$ is an $\mathcal{A}_X$ -quadratic form associated to an $\mathcal{A}_X$ -bilinear form b_X *iff*:

$$q_{E_U}(s) \equiv q_{E,U}(s) = b_U(s,s), \qquad \text{...........................[2.3]}$$

for any $s \in E_U \equiv E(U)$, with $U \subseteq X$ open, $b_U\colon E_U \rightarrow \mathcal{A}_U$ is an $\mathcal{A}_U$- quadratic form which satisfies the following:

$$a_V^U \, o \, q_{E_U} = q_{E_V} \, o \, e_V^U, \text{ with } V \subseteq U \text{ open} \qquad \text{...........................[2.4]}$$

Proposition 2.3 Let $\Gamma(E_X)$ be a complete presheaf, where E_X is a free $\mathcal{A}_X$ -module of rank n. For every open U in X, let $B(U)$ be the set consisting of all the bases of $E(U)$. If, for every U, V open in X with $V \subseteq U$, we have the restriction map $b_V^U\colon B(U) \rightarrow B(V)$, then the set $\Gamma(B_X) = (B(U), b_V^U)$, where $b_V^U = e_V^U/_{B(U)}$, is a complete presheaf.

Proof

(1). If U is an open subset of X and $(Ui)_{i\in I}$ is an open covering of U; let $s_1, s_2 \in E(U)$ such that $e_{Ui}^U(s_1) = e_{Ui}^U(s_2)$, for every $i\in I$. Suppose that $s_1, s_2 \in B(U)$ and $b_{Ui}^U(s_1) = b_{Ui}^U(s_2)$. Then, $b_{Ui}^U(s_1) = b_{Ui}^U(s_2)$, implies that $(e_{Ui}^U/_{B(U)})(s_1) = (e_{Ui}^U/_{B(U)})(s_2)$; i.e. $s_1 = s_2$ (the converse is certainly true).

(2). Let U and $(Ui)_{i\in I}$ be as in (1); moreover let $(s_i) \in \prod_i B(Ui)$ such that, for any $U_{ij} \equiv U_i \cap U_j \neq \emptyset$, in $(Ui)_{i\in I}$, we have: $b_{Uij}^{Ui}(si) = b_{Uij}^{Uj}(sj)$ which implies that $(e_{Uij}^{Ui}/_{B(Ui)})(si) = (e_{Uij}^{Uj}/_{B(Uj)})(sj)$.

Then, there exists an element $s \in B(U)$ such that $b_{Ui}^U(s) = s_i$, for all $i \in I$.

Hence shown that $\Gamma(B_X)$ is a complete presheaf.

Remark 2.4 Let E_X be a free $\mathcal{A}_X$-module (or a vector sheaf). If $q_{E_X}\colon E_X \rightarrow \mathcal{A}_X$ is an $\mathcal{A}_X$ -quadratic form associated to a symmetric $\mathcal{A}_X$ -bilinear form $b_X\colon E_X \oplus E_X \rightarrow \mathcal{A}_X$, then we have:

$$q_{E,U}(s)q_{E,U}(t) + q_{E,U}(t)q_{E,U}(s) = 2b_U(s,t), \qquad \text{........................[2.5]}$$

where $s, t \in E_U \equiv E(U)$, with $U \subseteq X$ open.

Definition 2.5 Let E_X be a free $\mathcal{A}_X$-module (or a vector sheaf). If $q_{E_X}\colon E_X \rightarrow \mathcal{A}_X$ is an $\mathcal{A}_X$ -quadratic form associated to an $\mathcal{A}_X$ -bilinear form $b_X\colon E_X \oplus E_X \rightarrow \mathcal{A}_X$, then the pair:

$$(E_X, q_{E_X}) = S\Gamma(E_X, q_{E_X}) \equiv (S\Gamma(E_X), q_{S\Gamma(E_X)}) \qquad \text{...........................[2.6]}$$

is called an $\mathcal{A}_X$ -quadratic space.

Definition 2.6 Let *X* be a topological space, Ω_X be a sheaf of (differential) $\mathcal{A}_X$ -modules over *X*, d_X be a derivative map as the $\mathbb{K}_X$ -sheaf morphism which is also $\mathbb{K}_X$-linear, where $\mathbb{K}_X$ = ($\mathbb{R}_X$ or $\mathbb{C}_X$) and $\mathcal{A}_X$ be a sheaf of unital $\mathbb{K}$-algebras over *X*, with $\mathbb{R} \equiv (\mathbb{R}, R, X)$ and $\mathbb{C} \equiv (\mathbb{C}, ¢, X)$ be respectively the sheaf of real numbers and the sheaf of complex numbers.

We define the triplet as treated in [12], and [14]:

$(\mathcal{A}_X , d_X , \Omega_X)$ ………………………….[2.7]

as the *differential triad* relative to $(X , \mathcal{A}_X)$ iff, for every *U,V* open in *X* with $V \subseteq U$, the Leibniz (product) rule:

$d_U(A.B) = A.d_U(B) + B.d_U(A)$ ………………………….[2.8]

is satisfied , with $A, B \in \mathcal{A}_U \equiv \mathcal{A}(U)$ and $d_U: \mathcal{A}_U \equiv \mathcal{A}(U) \to \Omega_U \equiv \Omega(U)$, be continuous and

$\mathbb{K}_U$-linear, and

$w_V^U \ o \ d_U = d_V \ o \ a_V^U$ ………………………….[2.9]

where $a_V^U : \mathcal{A}(U) \to \mathcal{A}(V)$ and $w_V^U : \Omega(U) \to \Omega(V)$ are restriction maps.

We set the differential triad as:

$dT_X = (\mathcal{A}_X , d_X , \Omega_X)$. ………………………….[2.10]

Definition 2.7 Let dT_X be a differential triad. If $(\mathcal{A}_X , q_{\mathcal{A}_X})$ and (Ω_X, q_{Ω_X}) are two $\mathcal{A}_X$-quadratic spaces, then $((\mathcal{A}_X , q_{\mathcal{A}_X}) , d_X , (\Omega_X, q_{\Omega_X}))$ is an *$\mathcal{A}_X$-quadratic differential triad* if , for any $\mathcal{A}_X$ −quadratic form $q_{\Omega_X} : \Omega_X \to \mathcal{A}_X$, there exists an endomorphism $q_{\mathcal{A}_X} : \mathcal{A}_X \to \mathcal{A}_X$ such that, for every *U, V* open in *X* with $V \subseteq U$, we have

$q_{\Omega_U} \ o \ d_U = q_{\mathcal{A}_U}$, $a_V^U \ o \ q_{\mathcal{A}_U} = q_{\mathcal{A}_V} \ o \ a_V^U$ and $a_V^U \ o \ q_{\Omega_U} = q_{\Omega_V} \ o \ w_V^U$ …………………………..[2.11]

Illustration 2.8 Let $\mathcal{A}_X = C_X^\infty$ and Ω_X^1 be, respectively, the sets of differential 0-forms and differential 1-forms over the smooth manifold *X*. If $(\mathcal{A}_X, d_X ,\Omega_X^1)$ is a differential triad on *X*, and $q_{\Omega_X^1} =: \Omega_X^1 \to \mathcal{A}_X$ is an $\mathcal{A}_X$ –quadratic form such that, for every *U* open in *X* and $w \in \Omega_U^1$ we have

$q_{\Omega_U^1}(w) = w^2$. Then, for any $a \equiv a(\alpha) = k\alpha$, with $k \neq 0$ and $k, \alpha \in \mathbb{K}_U$, there exists a map

$$q_{\mathcal{A}_U} : \mathcal{A}_U = C_U^\infty \to \mathcal{A}_U = C_U^\infty, \ a \to q_{\mathcal{A}_U}(a) = k^2 d\alpha^2.$$

After calculations, we find that: $q_{\mathcal{A}_U}(a) \equiv q_{\mathcal{A}_U}(a(\alpha)) = (q_{\Omega_U} \circ d_U)(a(\alpha)) = k^2 d\alpha^2.$

Note that the map $a : S_U \subseteq \mathcal{A}_{U_{\mathbb{K}_U}} \equiv (\mathcal{A}_{\mathbb{K}})_U \to \mathcal{A}_U \equiv C_U^\infty$ is such that $a(\alpha) = k\alpha$, for any $\alpha \in S_U$ and $k \in \mathbb{K}_U^*$, where S_U is a subset of the underlying of $\mathbb{K}_U$ in $\mathcal{A}_U$ denoted by $(\mathcal{A}_{\mathbb{K}})_U$.

Remark 2.9 According to the Definition 2.7 and the illustration 2.8, it follows that $q_{C_X^\infty}$ is a differential quadratic form. Thus, we use:

$qdT_X = ((C_X^\infty , q_{C_X^\infty}) , d_X , (\Omega_X, q_{\Omega_X}))$ ……………………………….[2.12]

as the quadratic differential triad relative to (X , C_X^∞).

Definitions 2.10 Let dT_{iX} and dT_{jX} be two quadratic differential triads relative to $(X , \mathcal{A}_{iX})$ and $(X , \mathcal{A}_{jX})$, respectively , with *i, j* = 1 ,2, …A *morphism of differential triads* from dT_{iX} to qdT_{jX} is a triplet

$(h_{\mathcal{A}_X}^{ij}, d_X^{ij} , h_{\Omega_X}^{ij})$ ……………………………………..[2.13]

such that:

$d_X^{ij}(h_{\mathcal{A}_X}^{ij}) \equiv S\Gamma(d_X^{ij})(h_{S\Gamma(\mathcal{A}_X)}^{ij}) = h_{S\Gamma(\Omega_X)}^{ij} \equiv h_{\Omega_X}^{ij}$ ……………………………….[2.14] and it satisfies the Leibniz (product) rule, given in [2.8] and [2.9] , where $h^{ij}_{\mathcal{A}_X} \in H^{ij}_{\mathcal{A}_X} = Hom_{\mathbb{K}_X}(\mathcal{A}_{iX}, \mathcal{A}_{jX})$ and $h^{ij}_{\Omega_X} \in H^{ij}_{\Omega_X} = Hom_{\mathcal{A}_X}(\Omega_{iX}, \Omega_{jX})$.

For every *U; V* open in *X* with $V \subseteq U$, we have: $W^U_V \; o \; d^{ij}_U = d^{ij}_V \; o \; A^U_V$

where $A^U_V : H^{ij}_{\mathcal{A}}(U) \to H^{ij}_{\mathcal{A}}(V)$ and $W^U_V : H^{ij}_{\Omega}(U) \to H^{ij}_{\Omega}(V)$ are restriction maps.

We have : - $h^{ij}_{\mathcal{A}_U}(1_{\mathcal{A}_{iU}}) = 1_{\mathcal{A}_{jU}}$),
if $1_{\mathcal{A}_{iU}}$ and $1_{\mathcal{A}_{jU}}$ are the units of $\mathcal{A}_{iU}$ and $\mathcal{A}_{jU}$, respectively, with *U* open in *X*;
- $h^{ij}_{\mathcal{A}_X}(1_x) = 1_x = \varinjlim_{x \in X} 1_{\mathcal{A}_j}(U) \equiv \varinjlim_{x \in X} 1_{\mathcal{A}_{jU}}$,
if 1_x denotes the unit of both $\mathcal{A}_{iX}$ and $\mathcal{A}_{jX}$, for all $x \in U \subseteq X$;
- $h^{ij}_{\Omega_X}(a_i \,.\, \omega_i) = h^{ij}_{\mathcal{A}_X}(a_i).\, h^{ij}_{\Omega_X}(\omega_i)$, for any $(a_i, \omega_i) \in \mathcal{A}_{iX} \times \Omega_{iX}$.

Notation 2.11 We denote *the category of differential triads* over *X* by:

$DiffT_X$ ……………………….[2.15]

where $DiffT_X \equiv (DiffT, \tau, X)$ and $\Gamma(DiffT) \equiv (DiffT(U), \tau^U_V)$ are, respectively, *sheaf and presheaf of categories of differential triads over* ***X*****,** with *U, V* opens in *X*.

Let Alg_X be *the category of* $\mathbb{K}_X$*-algebras over X*, $Diff_X$ be the set of differentials given in [2.7] and [2.13], all, over *X* and $DiffF_X$ be the *category of differential form over X*. We set:

$DiffT_X = (Alg_X, Diff_X, DiffF_X)$ ………………………..[2.16]

Definition 2.12 Let qdT_{iX} and qdT_{jX} be two quadratic differential triads relative to $(X, \mathcal{A}_{iX})$ and $(X, \mathcal{A}_{jX})$, respectively , with *i, j* = 1 ,2, …A *morphism of quadratic differential triads* from qdT_{iX} to qdT_{jX} is a triplet:

$(h^{ij}_{\mathcal{A}_X}, d^{ij}_X, h^{ij}_{\Omega_X})$, …………………………[2.17]

such that [2.13] , [2.14] and all properties of Definition 2.10 are satisfied. Also, we have

$h'^{ij}_{\mathcal{A}_X} \; o \; q_{\mathcal{A}_{iX}} = h^{ij}_{\mathcal{A}_X}$, $d_{ijX} \; o \; q_{\Omega_{iX}} = h^{ij}_{\Omega_X}$, $q_{\Omega_{iX}} \; o \; d_{iX} = q_{\mathcal{A}_{iX}}$ and $d_{jX} \; o \; h'^{ij}_{\mathcal{A}_X} = d_{ijX}$, ..………………………[2.18]

with $h'^{ij}_{\mathcal{A}_X}, h^{ij}_{\mathcal{A}_X} \in H^{ij}_{\mathcal{A}_X}$, $h^{ij}_{\Omega_X} \in H^{ij}_{\Omega_X}$, $q_{\mathcal{A}_{iX}} \in End(\mathcal{A}_{iX})$, $q_{\Omega_{iX}} \in Hom_{\mathbb{K}_X}(\Omega_{iX}, \mathcal{A}_{iX})$,

$d_{iX} \in Hom_{\mathbb{K}_X}(\mathcal{A}_{iX}, \Omega_{iX})$, $d_{jX} \in Hom_{\mathbb{K}_X}(\mathcal{A}_{jX}, \Omega_{jX})$ and $d_{ijX} \in Hom_{\mathbb{K}_X}(\mathcal{A}_{iX}, \Omega_{jX})$,

where d_{iX} , d_{jX} , d_{ijX} satisfy the Leibniz (product) rule, $q_{\mathcal{A}_{iX}}$ *and* $q_{\Omega_{iX}}$ are two quadratic forms and $h^{ij}_{\mathcal{A}_X}$, $h'^{ij}_{\mathcal{A}_X}$, $h^{ij}_{\Omega_X}$ are continuous maps.

We observe that the triplet: $((H^{ij}_{\mathcal{A}_X}, q_{H^{ij}_{\mathcal{A}_X}}), d^{ij}_X, (H^{ij}_{\Omega_X}, q_{H^{ij}_{\Omega_X}}))$

is a quadratic differential triad. For every *U, V* opens in *X* with $V \subseteq U$, we have:

$W^U_V \; o \; d^{ij}_U = d^{ij}_V \; o \; A^U_V.$

where $A^U_V : H^{ij}_{\mathcal{A}}(U) \to H^{ij}_{\mathcal{A}}(V)$ and $W^U_V : H^{ij}_{\Omega}(U) \to H^{ij}_{\Omega}(V)$ are restriction maps.

Notation 2.13 We denote the category of quadratic differential triads over *X* by:

$QDiffT_X$ ………………………[2.19]

where $QDiffT_X \equiv (QDiffT, \tau, X)$ and $\Gamma(QDiffT) \equiv (QDiffT(U), \tau^U_V)$ are, respectively, sheaf and presheaf of categories of quadratic differential triads over *X***,** with *U, V* open in *X*.

Let $QAlg_X$ be *the category of quadratic* $\mathbb{K}_X$*-algebra spaces over X,* $Diff_X$ be the set of differentials given in [2.7] and [2.13], all, over *X* and $QDiffF_X$ be the *category of quadratic differential forms* over *X*. We set:

$QDiffT_X = (QAlg_X, Diff_X, QDiffF_X)$. ……………………….[2.20]

Remarks 2.14

We can replace the topological space *X* by, respectively, the topological spaces $\mathrm{id_X}$ and $Hom(X, Y)$ to construct, respectively, the categories of quadratic differential triads over $\mathrm{id_X}$ and $Hom(X, Y)$**,** respectively, denoted by*:*

$QDiffT_{id_X}$ and $QDiffT_{Hom(X,Y)}$ ……………………….[2.21]

From sheaves over categories, we can replace the topological space *X*, respectively, by the categories $Open_X$ and *TOP* to construct, respectively, the categories of quadratic differential triads over $Open_X$ and *TOP***,** denoted:

$QDiffT_{Open_X}$ and $QDiffT_{TOP}$ ……………………….[2.22]

The categories $QDiffT_{id_X}$, $QDiffT_{Hom(X,Y)}$, and $QDiffT_{Open_X}$ are subcategories of the category

$QDiffT_{TOP} \equiv QDiffT$ ……………………….[2.23]

of quadratic differential triads over the category *TOP* of all topological spaces.

Let $(\Gamma(\mathcal{A}), \Gamma(d), \Gamma(\Omega))$ and $((h^{ij}_{\Gamma(\mathcal{A})}), \Gamma(d^{ij}), (h^{ij}_{\Gamma(\Omega)}))$ be, respectively, the complete presheaf and the complete presheaf morphism differential triads. Consider an operator *Q*, defined as follows:

$Q(\Gamma(\mathcal{A}), \Gamma(d), \Gamma(\Omega)) = ((\Gamma(\mathcal{A}), q_{\Gamma(\mathcal{A})}), \Gamma(d), (\Gamma(\Omega), q_{\Gamma(\Omega)}))$ ……………………….[2.24]

and :

$Q(\Gamma(h^{ij}_{\mathcal{A}}), \Gamma(d^{ij}), \Gamma(h^{ij}_{\Omega})) = (h^{ij}_{\Gamma(\mathcal{A})}, \Gamma(d^{ij}), h^{ij}_{\Gamma(\Omega)})$, ……………………….[2.25]

where $q_{\Gamma(\mathcal{A})}$ and $q_{\Gamma(\Omega)}$ satisfy the following condition :

$q_{\Gamma(\Omega)}\ o\ \Gamma(d) = q_{\Gamma(\mathcal{A})}$ ……………………….[2.26]

From the definition of the sheafification functor *S*, we have:

$Q(S\Gamma(\mathcal{A}), S\Gamma(d), S\Gamma(\Omega)) = ((\mathcal{A}, q_{\mathcal{A}}), d, (\Omega, q_{\Omega}))$, $Q(S\Gamma(h^{ij}_{\mathcal{A}}), S\Gamma(d^{ij}), S\Gamma(h^{ij}_{\Omega})) = (h^{ij}_{\mathcal{A}}, d^{ij}, h^{ij}_{\Omega})$.

We set :

$Q(\mathcal{A}, d, \Omega) = (((\mathcal{A}, q_{\mathcal{A}}), d, (\Omega, q_{\Omega})), Q(h^{ij}_{\mathcal{A}}, d^{ij}, h^{ij}_{\Omega}) = (h^{ij}_{\mathcal{A}}, d^{ij}, h^{ij}_{\Omega})$ ……………………….[2.27]

where $q_{\mathcal{A}}$ and q_{Ω} satisfy the condition :

$q_{\Omega}\ o\ d = q_{\mathcal{A}}$ ……………………….[2.28]

Here the question is: what kind of pairs $(q_{\mathcal{A}}, q_{\Omega})$ can satisfy the relation (2.28)?

We know that through the pair $(\mathcal{A}, \Omega)$, we can define several pairs of quadratic spaces $((\mathcal{A} = S\Gamma(\mathcal{A}), q^i_{S\Gamma(\mathcal{A})}), (\Omega = S\Gamma(\Omega), q^i_{S\Gamma(\Omega)}))$, where $i = 1,2,3,\ldots$ To answer this question, we have to fix a (differential) $\mathcal{A}$ -quadratic form $\dot{q}_{\Omega} : \Omega \rightarrow \mathcal{A}$ such that:

$\int \dot{q}_{\Omega} = \hat{q}_{\mathcal{A}}$ ……………………….[2.29]

where the symbol $\int\dot{q}_{\Omega}$ designs the "integral" of the differential form $\dot{q}_{\Omega}$, where $\hat{q}_{\mathcal{A}}$ is the primitive function of $\dot{q}_{\Omega}$. We have:

$\hat{q}_{\mathcal{A}} = q_{\mathcal{A}} + k$, with $k \in \mathcal{A}_{\mathbb{K}}$ ……………………….[2.30]

Note that $\mathcal{A}_{\mathbb{K}}$ designs the underlying of $\mathbb{K}$ in $\mathcal{A}$ and we realize an equivalence relation, ~ , defined in $End_{\mathbb{K}}(\mathcal{A}) \equiv Hom_{\mathbb{K}}(\mathcal{A}, \mathcal{A})$ as follows:

$q^i_{\mathcal{A}} \sim q^j_{\mathcal{A}}$ *iff* $q^j_{\mathcal{A}} - q^i_{\mathcal{A}} = k$, with $k \in \mathcal{A}_{\mathbb{K}}$ ……………………….[2.31]

Referring to the above concern , the (differential) $\mathcal{A}$ -quadratic form $\dot{q_\Omega} : \Omega \rightarrow \mathcal{A}$ that is defined, leads us to choose a subcategory $\dot{Q}DiffT$ of *QDiffT* whose objects

$$((\mathcal{A}, \hat{q}_{\mathcal{A}}), d, (\Omega, \dot{q_\Omega})) \qquad \text{[2.32]}$$

satisfy the relations [2.28] [2.29] and [2.30], for any pair $(\mathcal{A}, \Omega)$.

Definition 2.16 Let us denote by $\hat{Q}_{\mathcal{A}}$ and $\dot{Q_\Omega}$ be the set of differential quadratic forms $\dot{q_\Omega} : \Omega \rightarrow \mathcal{A}$,… and the set of quadratic forms $\hat{q}_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{A}$, …, respectively which satisfy [2.28] ,[2.29] and[2.30].

We define the triplet: $(\dot{Q_\Omega}, \smallint, \hat{Q}_{\mathcal{A}})$ ……………………………..[2.33]

where $\int : \dot{Q_\Omega} \rightarrow \dot{Q_{\mathcal{A}}}, \dot{q_\Omega} \rightarrow \int \dot{q_\Omega} = \hat{q}_{\mathcal{A}}$ is continuous and satisfies [2.30]. We say that the triplet [2.33] is *the quadratic integral triad* over $\mathcal{A}$ if and only if, for any $x \in U$, with U open in X, we have

$$\smallint_x = \varinjlim_{x\epsilon U} \smallint_U = \varinjlim_{x\epsilon U} (d_U)^{-1} = (d_x)^{-1} , \qquad \text{[2.34]}$$

where d satisfies [2.8] [2.9] and [2.10]. We set:

$$QIT = (\dot{Q_\Omega}, \smallint, \hat{Q}_{\mathcal{A}}) = (S\Gamma(\dot{Q_\Omega}), S\Gamma(\smallint), S\Gamma(\hat{Q}_{\mathcal{A}})) \equiv S\Gamma\,(\dot{Q_\Omega}, \smallint, \hat{Q}_{\mathcal{A}}) = S\Gamma(Q\,IT) \qquad \text{[2.35]}$$

The above expression represents the *category of quadratic integral triads*.

Theorem 2.17 Let $M \in \mathcal{A}^{n\times m}$ be a matrix, $\Omega = \mathcal{A}^{1\times m} / \mathcal{A}^{1\times m}M$ be the left $\mathcal{A}$-module finitely represented by M and $p : \mathcal{A}^{1\times m} \rightarrow \Omega$ be the canonical projection onto Ω. If , for any open U in X, $\{a_i\}$ is the standard $\mathcal{A}(U)$-basis of $\mathcal{A}^{1\times m}(U) \equiv \mathcal{A}(U)^{1\times m}$, $m_i = p_U(a_i)$, with $i=1,...,n$, and E be a left $\mathcal{A}$-module , then we have the following abelian group $\mathcal{A}(U)$-isomorphism :

$$Hom_{\mathcal{A}_U}(\Omega_U, E_U) \rightarrow Ker_{E_U}(\widehat{M_U}), f_U \rightarrow e_U ,$$

where $\widehat{M_U} \in End_{\mathcal{A}_U}(E_U^n)$ and $\{e \in E_U^n / Me = 0\}$.

Proof. It is obvious that, because there is a one to one correspondence between the elements of $Hom_{\mathcal{A}_U}(\Omega_U, E_U)$ and $Ker_{E_U}(\widehat{M_U})$.(Refer also to [19], Theorem 1.1.1.).

III. Quadratic functorial operator

Some notions of this section are treated classically in [6] and [8], but in Abstract Differential Geometry in [6], [12] and [14].

Definition 3.1 Let $\dot{Q} : DiffT \rightarrow \dot{Q}DiffT \subseteq QDiffT$ be an operator which satisfies the relations [2.26], and [2.27]. Then $\dot{Q}$ is a quadratic functorial operator .

Theorem 3.2 The quadratic functorial operator $\dot{Q}$ is a covariant functor.

Proof. Consider $dT_i, dT_j, dT_k \in Ob(DT)$ and $mdT_{ij} \in Hom(dT_i, dT_j)$, $mdT_{ik} \in Hom(dT_i, dT_k)$ and $mdT_{jk} \in Hom(dT_j, dT_k)$. If we apply the operator $\dot{Q}$ on *DiffT*, we have:

(1) $\dot{Q}(mdT_{jk} \circ mdT_{ij}) = m\dot{Q}dT_{jk} \circ m\dot{Q}dT_{ij} = \dot{Q}(mdT_{jk}) \circ \dot{Q}(mdT_{ij})$,
(2) $\dot{Q}(id_{dT_i}) = id_{\dot{Q}dT_i} = id_{\dot{Q}(dT_i)}$.

Remark 3.3 By convenience, we set:

$$\dot{Q}DiffT := <(DiffT, \dot{q} := \{\hat{q}_{\mathcal{A}}, \dot{q}_\Omega\})>, \qquad \text{[3.1]}$$

where $\{\dot{q}_{\mathcal{A}}, \dot{q}_\Omega\}$ satidfies the relations from [2.25] up [2.32] .

We construct the quadratic functor operators as follows:

$\dot{Q}_X : DiffT_X \rightarrow \dot{Q}DiffT_X$, $\dot{Q}_{Hom(X,Y)} : DiffT_{Hom(X,Y)} \rightarrow \dot{Q}DiffT_{Hom(X,Y)}$, $\dot{Q}_{Open_X} : DiffT_{Open_X} \rightarrow \dot{Q}DiffT_{Open_X}$,
$\dot{Q}_{TOP} : DiffT_{TOP} \rightarrow \dot{Q}DiffT_{TOP}$.

Let $\dot{\mathbb{Q}}_{\mathcal{A}}$ and $\dot{\mathbb{Q}}_\Omega$ be the matrices representing the quadratic forms $\dot{q_{\mathcal{A}}}$ and $\dot{q_\Omega}$. Thus, for any $S = (s^1, s^2, ..., s^n)$ in $Gl(n, \mathcal{A}_U)$ and $R = (r^1, r^2, ..., r^m)$ in $M_m(\Omega_U)$, with $U \subsetneq X$ open, we have

$S\,\dot{\mathbb{Q}}_{\mathcal{A}_U}\tilde{S} = 0, \qquad R\,\dot{\mathbb{Q}}_{\Omega_U}\tilde{R} = 0$ ……………………………..[3.2]

where $\tilde{S}$ and $\tilde{R}$ are transposed of S and R, respectively.

Remark 3.4 If (β_X , q_{β_X}) is a subalgebra of $(\mathcal{A}_X, q_{\mathcal{A}_X})$; then, using the duality notion, there exists a subspace of $(\mathcal{A}_X, q_{\mathcal{A}_X})$, said , $(\beta_X , q_{\beta_X})^{\perp}$ such that:

$(\beta_X , q_{\beta_X}) \oplus (\beta_X , q_{\beta_X})^{\perp} = (\mathcal{A}_X, q_{\mathcal{A}_X})$ ……………………………….[3.3]

and consequently we have the following lemma and theorem, where ${\beta_X}^{\perp}$ designs the orthogonal of β_X and we set:

$(\beta_X , q_{\beta_X})^{\perp} \equiv ({\beta_X}^{\perp}, q_{{\beta_X}^{\perp}})$ ……………………………….[3.4]

Lemma 3.5 Let $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$ and $\Omega^1 = {\Omega_1}^1 \oplus {\Omega_2}^1$ such that $dT_1 = (\mathcal{A}_1 , d , {\Omega_1}^1)$ and $dT_2 = (\mathcal{A}_2 , d , {\Omega_2}^1)$ are differential triads. Then , there exists a differential triad $dT = S\Gamma(dT) = (\mathcal{A}, d , \Omega^1)$ such that

$dT = dT_1 \oplus dT_2$ ……………………………….[3.5]

Proof. The *existence* results to the following decompositions:

Let $a = a_1 + a_2$ and $s = s_1 + s_2$ be decompositions of sections a $\epsilon\, \mathcal{A}_U$, with $(a_1 , a_2) \,\epsilon\, {\mathcal{A}_1}_U \oplus \mathcal{A}_{2U}$, and $s \,\epsilon\, {\Omega^1}_U$, with $(s_1 , s_2) \,\epsilon\, {{\Omega_1}^1}_U \oplus {{\Omega_2}^1}_U$, such that $\mathcal{A}_U$ and ${\Omega^1}_U$ verify respectively, the splitting $\mathcal{A}_U = {\mathcal{A}_1}_U \oplus \mathcal{A}_{2U}$ and ${\Omega^1}_U = {{\Omega_1}^1}_U \oplus {{\Omega_2}^1}_U$, $U \subseteq X$ open. Then, we have the map $d_U : {\mathcal{A}_1}_U \oplus \mathcal{A}_{2U} \rightarrow {{\Omega_1}^1}_U \oplus {{\Omega_2}^1}_U$, such that $d_U\,(a_1 + a_2) = d_U\,(a_1) + d_U\,(a_2) \equiv s_1 + s_2$ and $d_U\,(a_1 . a_2) = d_U\,(a_1)\, a_2 + a_1\; d_U\,(a_2)$. We obtain

$dT \equiv (\mathcal{A}, d, \Omega^1) = (\mathcal{A}_1 \oplus \mathcal{A}_2, d_1 + d_2, {\Omega_1}^1 \oplus {\Omega_2}^1) = (\mathcal{A}_1 , d , {\Omega_1}^1) \oplus (\mathcal{A}_2 , d , {\Omega_2}^1) = dT_1 \oplus dT_2$.

Theorem 3.6 Let $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$ and and $\Omega^1 = {\Omega_1}^1 \oplus {\Omega_2}^1$such that $qdT_1 = ((\mathcal{A}_1, q_{\mathcal{A}_1}) , d, ({\Omega_1}^1 , q_{{\Omega_1}^1}))$ and $qdT_2 = ((\mathcal{A}_2, q_{\mathcal{A}_2}) , d ,({\Omega_2}^1 , q_{{\Omega_2}^1}))$ be two differential triads. Then, there exists a differential triad $qdT = ((\mathcal{A}, q_{\mathcal{A}}), d, (\Omega^1, q_{\Omega^1}))$ such that:

$qdT = qdT_1 \oplus qdT_2$ …………………………..[3.6]

Proof. The existence results to the decompositions of $\mathcal{A}$ and Ω^1in direct sums. Also, using *Lemma* 3.5 and the definition of a quadratic differential triad, we have, for all $U \subseteq X$ open,

$qdT_U \quad = qdT_{1U} \oplus qdT_2U.$

Remarks 3.7 -Using the relation [2.28] and Theorem 3.6, we observe that $q_{{\Omega_1}^1} \circ d_1 = q_{\mathcal{A}_1}$ and $q_{{\Omega_2}^1} \circ d_2 = q_{\mathcal{A}_2}$ imply that $q_{\Omega^1} \circ d = q_{\mathcal{A}}$.

-Let $H^{ij}_{\mathcal{A}} = Hom_{\mathbb{K}}(\mathcal{A}_i, \mathcal{A}_j)$ and $H^{ij}_{\Omega^1} = Hom_{\mathcal{A}}({\Omega_i}^1 , {\Omega_j}^1)$ be two sheaves such that the triplet $(H^{ij}_{\mathcal{A}}, d^{ij}, H^{ij}_{\Omega^1})$ is a differential triad over X, id_X, $Hom(X ,Y)$, $Open_X$ or *TOP*. Then, the quadratic functorial operator $\dot{Q}$ applies on the differential triad $dT^{ij} = (H^{ij}_{\mathcal{A}}, d^{ij}, H^{ij}_{\Omega^1})$ so that this differential triad becomes the quadratic differential triad:

$\dot{Q}(dT^{ij}) = \dot{q}dT^{ij}$ ………………………[3.7]

IV. Homology – cohomology-resolutions

Some notions of this section are treated classically in [2-4], [6], [8], [10], [17-22], [26] and [27], in Abstract Algebra in [6] and in Abstract Differential Geometry in [12-14].

Definition 4.1 A complex of free left (resp. right) $\mathcal{A}$-modules denoted by

$\Omega^* \equiv \;\ldots \xrightarrow{d^{i-1}} \Omega^i \xrightarrow{d^i} \Omega^{i+1} \xrightarrow{d^{i+1}} \ldots$ …………………………..[4.1]

is a sequence of left (resp. right) $\mathcal{A}$-homomorphisms $S\Gamma(d^i)$: $S\Gamma(\Omega^i) \rightarrow S\Gamma(\Omega^{i+1})$ between left (resp. right) $\mathcal{A}$-modules which satisfy, for any open U in X, :

$Im\, d^{i-1}(U) \subseteq Ker\, d^i(U)$, i.e., ${d_U}^{\,i} \circ {d_U}^{\,i-1} = 0_U \equiv 0$, for all $i \in Z$. …………………………[4.2]

We set:

$d^0 = \partial, d^i = d,$ for any i ≥ 1, …………………………[4.3]

where the symbol *d* designs the differential $\mathcal{A}$-homomorphism.

If we specify the order of the set Ω= $S\Gamma(\Omega)$ of differential forms (sheaf of differential $\mathcal{A}$-modules) by setting:

$\Omega^0 = \mathcal{A}$ and $\Omega^i \equiv (\Omega^1)^i = \Lambda^i \Omega^1$, …………………………[4.4]

with $\wedge \equiv \wedge_{\mathcal{A}}$ be the *exterior* (the *skew symmetric homological tensor) product*, for any $i \geq 1$.

We have, more explicitly : $\Omega^1 = \mathcal{A} \wedge \Omega$, $\Omega^2 = \mathcal{A} \wedge \Omega^1 \wedge \Omega^1$, ...

Definitions 4.2 The defect of exactness of [4.1] is left (respectively right) $\mathcal{A}$-module defined by

$H^i(\Omega^*) = Ker\ d^i / Im\ d^{i-1}$ …………………………[4.5]

The complex [4.1] is said to be exactness of [4.1] at Ω^i if the left (resp. right) $\mathcal{A}$-module defined by

$H^i(\Omega^*) = S\Gamma(H^i(\Omega^*)) = S\Gamma(0) = 0$ …………………………..[4.6]

In other words $Im\ d^{i-1} = S\Gamma(Im\ d^{i-1}) = S\Gamma(Ker\ d^i) = Ker\ d^i$.

The complex [4.1] is said to be *exact* if: $Ker\ d^i = Im\ d^{i-1}$, for all $i \in z$ ……………………[4.7]

Definition 4.3 A complex of free left (resp. right) $\mathcal{A}$-modules denoted by

$\Omega_* \equiv \ \dots \xrightarrow{\int_{i+2}} \Omega_{i+1} \xrightarrow{\int_{i+1}} \Omega_i \xrightarrow{\int_i} \dots$ ………………………[4.8]

is a sequence of left (resp. right) $\mathcal{A}$-homomorphisms $\iint_{i+1} : \Omega_{i+1} \rightarrow \Omega_i$ between left (resp. right) $\mathcal{A}$-modules which satisfy, for any open *U* in *X*:

$Im \int_{i+1}(U) \subseteq Ker \int_i(U)$, for all $i \in Z$. ……………………………[4.9]

Also, for any $x \in U$, with *U* open in *X* :

$\int_{ix} = \varinjlim_{x \in U} \int_{iU} = \varprojlim_{x \in U} (d_U{}^i)^{-1} = (d_x{}^i)^{-1}$, for any $i \geq 1$, …………………………...[4.10]

to the nearest constant , where the symbol ∫ designs the *integral* $\mathcal{A}$*-homomorphism.*

The defect of exactness of [4.8] at Ω_i is the left (resp. right) $\mathcal{A}$-module defined by

$H_i(\Omega_*) = S\Gamma(H_i(\Omega_*)) = S\Gamma(Ker \int_i / Im \int_{i+1}) = S\Gamma(Ker \int_i) / S\Gamma(Im \int_{i+1}) = Ker \int_i / Im \int_{i+1}$ ……………………………[4.11]

The complex [4.8] is said to be *exact* at Ω_i if:

$H_i(\Omega_*) = S\Gamma(H_i(\Omega_*)) = S\Gamma(0) = 0$ …………………………...[4.12]

In other words $Im \int_{i+1} = S\Gamma(Im \int_{i+1}) = S\Gamma(Ker \int_i) = Ker \int_i$.

The complex [4.8] is *exact iff*:

$Ker \int_i = Im \int_{i+1}$, for all $i \in Z$ …………………………..[4.13]

Remark 4.4 The complexes [4.1] and [4.8] are represented as follows:

$$\begin{array}{ccccccccc} \Omega^* & \equiv & \dots & \xrightarrow{d^{i-1}} & \Omega^i & \xrightarrow{d^i} & \Omega^{i+1} & \xrightarrow{d^{i+1}} & \dots \\ & & & & \| & & \| & & \\ \Omega_* & \equiv & \dots & \xleftarrow{\int_i} & \Omega_i & \xleftarrow{\int_{i+1}} & \Omega_{i+1} & \xleftarrow{\int_{i+2}} & \dots , \end{array}$$

where we set $\Omega^i = \Omega_i$, within $\mathcal{A}$-isomorphism, for all $i \in z$.

In other words, we have *:*
$\int_{i+1} o\ d^i = id_{\Omega^i}$ ……………………….[4.14]

Definitions 4.5 Referring to [4.5] and [4.11] , we define $H^i(\Omega^*)$ and $H_i(\Omega_*)$ as respectively *the i^{th} cohomology and the i^{th} homology.*

Remark 4.6 If we replace, in the Remark 4.4**,** Ω by the differential triad dT, then we obtain the isomorphism between the complex of differential triads and complex of integral triads, through the following diagram:

$$\begin{array}{ccccccccc} dT^* & \equiv & \dots & \xrightarrow{mdT^{i-1}} & dT^i & \xrightarrow{mdT^i} & dT^{i+1} & \xrightarrow{mdT^{i+1}} & \dots \\ & & & & \| & & \| & & \\ dT_* & \equiv & \dots & \xleftarrow{m\int T_i} & \int T_i & \xleftarrow{m\int T_{i+1}} & \int T_{i+1} & \xleftarrow{m\int T_{i+2}} & \dots , \end{array}$$

where we set $dT^i = \int T_i$, within $\mathcal{A}$-isomorphism to the nearest constant, and mdT^i and $m\int T_i$ two morphisms, respectively, of differential triads and of integral triads, for all $i \in Z$, with *dT= SΓ(dT) , mdT= SΓ(mdT), ∫T= SΓ(∫T)* and *m∫T= SΓ(m∫T).* We observe that $dT = (\mathcal{A}, d, \Omega)$ *and* $\int T = (\Omega, \int, \mathcal{A})$ are *differential triad* and *integral triad,* respectively.

In other words, we have *:*

$m\int T_{i+1}\ o\ mdT^i = id_{dT^i}$ ………………………….[4.15]

Definitions 4.7 Referring to [4.5], [4.11] and Definitions 4.5, we define $H^i(dT^*)$ and $H_i(\int T_*)$ as, respectively, the i^{th} co homology and the *i^{th} homology* of differential and integral triads.

Definitions 4.8 A *finite free resolution* of the left $\mathcal{A}$-module Ω is an exact sequence of the form *(see [19], Definition 1.2.1, p.13-14)*

$$\dots \xrightarrow{\hat{L}_{2x}} \mathcal{A}_x^{1\times r_1} \xrightarrow{\hat{L}_{1x}} \mathcal{A}_x^{1\times r_0} \xrightarrow{\pi_x} \Omega_x \to o_x , \qquad x \in X \qquad \text{………………………..[4.16]}$$

where $\hat{L}_{ix}: \mathcal{A}_x^{1\times r_i} \to \mathcal{A}_x^{1\times r_{i-1}}$ is the left $\mathcal{A}_x$-homomorphism defined as follows :

$$\hat{L}_{ix}(\sigma) = \underset{\overleftarrow{x \in U}}{lim} \hat{L}_{iU}(\sigma) = \underset{\overleftarrow{x \in U}}{lim}\ \sigma L_{iU} = \sigma L_{ix} , \qquad x \in U \subseteq X \qquad \text{…………………………..[4.17]}$$

for all $\sigma_x \in \mathcal{A}_x^{1\times r_i}$, with $L_{ix} \in \mathcal{A}_x^{r_i \times r_{i-1}}$ *or* $L_{ix} \in \mathcal{A}_x^{r_i \times r_{i-2}}$...

A *finite free resolution of the right $\mathcal{A}$-module* Ω is an exact sequence such that , for any $x \in U$, with U in X, we have

$$0 \leftarrow \Omega_x \xleftarrow{\tau_x} \mathcal{A}_x^{s_0\times 1} \xleftarrow{\hat{R}_{1x}} \mathcal{A}_x^{s_1\times 1} \xleftarrow{\hat{R}_{2x}} \dots , \qquad x \in X \qquad \text{………………………….[4.18]}$$

where $\hat{R}_{ix}: \mathcal{A}_x^{1\times s_i} \to \mathcal{A}_x^{1\times s_{i-1}}$ is the left $\mathcal{A}_x$-homomorphism defined as follows :

$$\hat{R}_{ix}(\rho) = \underset{\overleftarrow{x \in U}}{lim} \hat{R}_{iU}(\rho) = \underset{\overleftarrow{x \in U}}{lim}\ \rho R_{iU} = \rho R_{ix} , \qquad x \in U \subseteq X \qquad \text{………………………….[4.19]}$$

for all $\rho_x \in \mathcal{A}_x^{1\times s_i}$, with $R_{ix} \in \mathcal{A}_x^{s_{i-1}\times s_i}$ or $R_{ix} \in \mathcal{A}_x^{s_{i-2}\times s_i}$…

V. Applications

V.I. Let us consider the first set of Maxwell equations, said,

$$\begin{cases} \partial_t B + (\nabla \wedge E) = 0 \\ \nabla \cdot B = o \end{cases} \qquad \text{…………………………..[5.1]}$$

where B(resp. E) designs the magnetic(resp. electric) field, 0(resp. o) designs the zero vector (resp. the zero scalar) and $\mathcal{A} = \mathbb{Q}[\partial_t, \partial_1, \partial_2, \partial_3]$ be the polynomial algebra of operators with rational constant coefficients, with $\partial_i = \frac{\partial}{\partial_i} \equiv \frac{\partial}{\partial y_i}$,i=0,1,2,3 such that $\frac{\partial}{\partial y_0} = \frac{\partial}{\partial t}$.

If $R_1 \in \mathcal{A}^{4\times 6}$ is the representation matrix of [5.1], it follows that, for any $x \in X$, we have:

$$R_{1x}= \begin{pmatrix} \partial_t & 0 & 0 & 0 & -\partial_3 & \partial_2 \\ 0 & \partial_t & 0 & \partial_3 & 0 & -\partial_1 \\ 0 & 0 & \partial_t & -\partial_2 & \partial_1 & 0 \\ \partial_1 & \partial_2 & \partial_3 & 0 & 0 & 0 \end{pmatrix}_x$$

Let $\Omega = \mathcal{A}^{1\times 6}/\ \mathcal{A}^{1\times 4}R_1$ be the left $\mathcal{A}$-module finitely represented by R_1 (see Theorem 2.17). Using Definition 4.8 and [5.1], we observe that the left $\mathcal{A}$-module Ω admits, for any $x \in X$, the following free resolution:

$$o_x \rightarrow \mathcal{A}_x \xrightarrow{\hat{L}_{2x}} \mathcal{A}_x{}^{1\times 4} \xrightarrow{\hat{L}_{1x}} \mathcal{A}_x{}^{1\times 6} \xrightarrow{\pi_x} \Omega_x \rightarrow o_x, \qquad \text{...[5.2]}$$

where the matrix $R_{2x}=(\partial_1, \partial_2, \partial_3, -\partial_t)_x \in \mathcal{A}_x{}^{1\times 4}$ defines the compatibility conditions

$$(\nabla \cdot J - \partial_t I)_x = o_x \qquad \text{...[5.3]}$$

of the inhomogeneous linear system

$$\begin{cases} (\partial_t B + (\nabla \wedge E))_x = J_x \\ (\nabla \cdot B)_x = \rho_x \end{cases} \qquad \text{...[5.4]}$$

where *B, E, J* are vectors and ρ is a scalar. In this case, J is the density of current and ρ is the charge.

If *\$= SΓ(\$)* is a open convex subsheaf of $\mathcal{A}_{\mathbb{R}}{}^4$, where $\mathcal{A}_{\mathbb{R}}$ is real the underlying of $\mathcal{A}$, then the space C^∞*(\$)* of smooth functions on *\$* is an injective $\Omega = \mathcal{A}_{\mathbb{R}}[\partial_1, \partial_2, \partial_3, \partial_4]$-module. Using the formula [5.4] , we can easily check, through the parameterization, that , for any $x \in X$, we can set:

$$\begin{cases} B_x = (\nabla \wedge A)_x \\ E_x = (-\partial_t A - \nabla v)_x , \end{cases} \qquad \text{...[5.5]}$$

where $A \in F^3$ and $v \in F$, with F be an injective $\mathcal{A}_{\mathbb{R}}[\partial_1, \partial_2, \partial_3, \partial_4]$-module C^∞(\$). We observe that *(A,v)* is *the quadric-potential* of [5.1] or [5.4], i.e., $\hat{R}_1 = R_0 F^4$ in the exact sequence:

$$0_x \leftarrow F_x \xleftarrow{\hat{R}_{3x}} F_x{}^3 \xleftarrow{\hat{R}_{2x}} F_x{}^2 \xleftarrow{\hat{R}_{1x}} F_x \xleftarrow{\hat{R}_0 x} Hom_{\mathcal{A}_{\mathbb{R}x}}(\Omega_x, F_x) \leftarrow 0_x. \qquad \text{...[5.6]}$$

Note that the quadric-potential *(A,v)* is not uniquely defined since the right-hand side of [5.5] is parameterized, for example, by :

$$\begin{cases} A_x = (-\nabla\varepsilon)_x \\ v_x = (\partial_t \varepsilon)_x , \end{cases} \qquad \text{...[5.7]}$$

in other words,

$$Ker_F(\hat{R}_0)= R_{-1}F.$$

Consequently, for any $\varepsilon \in F$, the gauge transformations $A \rightarrow A - \nabla\varepsilon$, $v \rightarrow v + \partial_t\varepsilon$ (See [12] and [19]) give the same fields E and B.

V.II. Consider the electromagnetism Lagrangian functional, given by:

$$\iint \tfrac{1}{2}(\epsilon_0\|E\| - \tfrac{1}{\mu_0}\|B\|)\, dt\, dy_1 dy_2 dy_3, \qquad \text{...[5.8]}$$

where $\epsilon_0(resp.\, \mu_0)$ is the dielectric constant(resp. the magnetic constant), under the differential constraint formed by the first set of Maxwell's equations [5.1] . By varying the Lagrangian functional, we introduce , for any $x \in X$, the maps

$$\sigma_x\colon F_x{}^6 \rightarrow F_x{}^6, \begin{pmatrix} B \\ E \end{pmatrix}_x \rightarrow \begin{pmatrix} -\frac{1}{\mu_0}B \\ \epsilon_0 E \end{pmatrix}_x \qquad \text{...[5.9]}$$

and

$$(\tau \circ \sigma)_x : F_x^{\,6} \to F_x^{\,4},\ (B,E)_x \to \begin{cases} \partial_t B + (\nabla \wedge E))_x = 0_x \\ (\nabla \cdot B)_x = 0_x \\ (\frac{1}{\mu_0}(\nabla \wedge E) - \epsilon_0\ \partial_t E)_x = 0_x \\ (\epsilon_0 \nabla \cdot B)_x = 0_x \end{cases} \quad [5.10]$$

which is the complete set of Maxwell equations. Using [4.24], we can present the components of *B*(resp.*E*) by :

$(\frac{1}{{c_0}^2}{\partial_t}^2 - \Delta)B_i$= o (resp. $(\frac{1}{{c_0}^2}({\partial_t}^2 - \Delta)E_i$= o), ……………………………[5.11]

where $\Delta = {\partial_1}^2 + {\partial_2}^2 + {\partial_3}^2$ is the Laplacian operator and ${c_0}^2 = \frac{1}{\mu_0 \epsilon_0}$, i.e. the fields *B* and *E* are space- time waves.
Now, consider the polynomial operator $A :\ F^4 \to F^4$ obtained by substituting the previous parametrization into the last two equations of [5.5] and by using the relation

$\nabla \wedge \nabla \wedge A = \nabla(\nabla . A) - \Delta A.$

We obtain, for any $x \in X$:

$$(A,v) \to \begin{cases} \frac{1}{\mu_0}(\frac{1}{{c_0}^2}\frac{\partial^2 A}{\partial t^2} - \Delta A + \nabla\left(\nabla . A + \frac{1}{{c_0}^2}\ \frac{\partial v}{\partial t}\right))_x = J_x \\ (\epsilon_0(\frac{1}{{c_0}^2}\frac{\partial^2 v}{\partial t^2} - \Delta v - \frac{\partial v}{\partial t}\left(\nabla . A + \frac{1}{{c_0}^2}\ \frac{\partial v}{\partial t}\right))_x = \rho_x \end{cases} \quad ………………………….....[5.12]$$

Then, putting together equations [5.5] and [5.11], and using [1.12], we obtain the system

$$\begin{cases} \frac{1}{\mu_0}(\frac{1}{{c_0}^2}\frac{\partial^2 A}{\partial t^2} - \Delta A + \nabla(\nabla . A + \frac{1}{{c_0}^2}\ \frac{\partial v}{\partial t}) = J \\ \epsilon_0(\frac{1}{{c_0}^2}\frac{\partial^2 v}{\partial t^2} - \Delta v - \frac{\partial v}{\partial t}(\nabla . A + \frac{1}{{c_0}^2}\ \frac{\partial v}{\partial t}) = \rho \\ \nabla \wedge A = B \\ -\partial_t A - \nabla v = E \end{cases} \quad …………………………...[5.13]$$

In electromagnetism, by fixing the Lorentz gauge defined by $\nabla . A + \frac{1}{{c_0}^2}\ \frac{\partial v}{\partial t} = 0$ the system [5.13] is generally simplified as follows

$$\begin{cases} \frac{1}{\mu_0}\left(\frac{1}{{c_0}^2}\frac{\partial^2 A}{\partial t^2} - \Delta A\right) = J \\ \epsilon_0\left(\frac{1}{{c_0}^2}\frac{\partial^2 v}{\partial t^2} - \Delta v\right) = \rho \\ \nabla \wedge A = B \\ -\partial_t A - \nabla v = E \end{cases} \quad ……………………………… [5.14]$$

This result shows us that each component of the quadri-potential *(A,v)* is a space-time wave, where

(A,v) $= \bigcup_{x \in X}(A,v)_x \equiv \sum_{x \in X}(A,v)_x$.

VI. Conclusion and Future Development

In this article, we have constructed categories QDT_X, QDT_{id_X} and $QDT_{Hom(X,Y)}$ whose objects are quadratic differential triads qdT_i = $((\mathcal{A}_i, q_{\mathcal{A}_i}), d_i, ({\Omega^1}_i, q_{{\Omega^1}_i}))$ and whose morphisms $mqdT^{ij} = (h^{ij}_{\mathcal{A}}, d^{ij}, h^{ij}_{\Omega^1})$ satisfy [2.18], respectively, over some topological spaces *X*, id_X and *Hom(X , Y).* We have generalize these concepts over the categories $Open_X$ and *TOP* to construct categories denoted by QDT_{Open_X} and QDT_{TOP}, or in the particular cases (see Definition 3.1 and Remark 3.3) by $\dot{Q}DT_{Open_X}$ and $\dot{Q}DT_{TOP}$, respectively. We have also introduced the notion of integral triad. The notions of (quadratic) differential triads and (quadratic) integral triads allowed us to analyze respectively the co homology and homology concepts. We have finished our paper with some physics applications in Electromagnetism. These notions can be applied for future research when dealing with The Poincaré gauge (or radial gauge), Weyl gauge known as temporal gauge or Hamiltonian gauge, axial gauge and gauge fixing.

To conclude, one should note that many aspects of this peculiar and techniques still remain to be explored. For instance, it would be interesting to study:

- The *2-Minkowsky* differential triads that we defined as an image of the 2-real *Euclidian* differential triad through a functorial morphism.
- The *Clifford* connection triad algebras with physics applications.

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