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# RESEARCH ARTICLE

## INTEGRAL SOLUTIONS OF THE SEXTIC EQUATION WITH THREE UNKNOWNS $(4k-1)(x^2+y^2)-(4k-2)xy=4(4k-1)z^6$

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### ABSTRACT

The sextic non-homogeneous equation with four unknowns represented by the Diophantine equation is$(4k-1)(x^2+y^2)-(4k-2)xy=4(4k-1)z^6$ analyzed for its patterns of non-zero distinct integral solutions are illustrated. Various interesting relations between the solutions and special numbers, namely polygonal numbers, Pyramidal numbers, Jacobsthal numbers, Jacobsthal-Lucas number are exhibited.

***Key Words:*** Integral solutions, Sextic, Non-homogeneous equation.

## INTRODUCTION

The theory of Diophantine equations offers a rich variety of fascinating problems (Dickson, 1952; Mordell, 1969; Telang, 1996; Carmichael, 1959). Particularly, in (Gopalan and Sangeetha, 2010; Gopalan *et al.,* 2007; Gopalan *et al.,* 2010), sextic equations with three unknowns are studied for their integral solutions. (Gopalan and Vijaya Sankar, 2010; Gopalan *et al.,* 2013; Gopalan *et al.,* 2012; Gopalan *et al.,* 2012; Gopalan *et al.,* 2013; Gopalan *et al.,* 2013; Gopalan *et al.,* 2013) analyze sextic equations with four unknowns for their non-zero integer solutions (Gopalan *et al.,* 2013; Gopalan *et al.,* 2012; Gopalan *et al.,* 2014) analyze sextic equations with five unknowns for their non-zero solutions. This communication concerns with yet another interesting a non-homogeneous sextic equation with three unknowns given Infinitely many non-zero inte $(4k-1)(x^2+y^2)-(4k-2)xy=4(4k-\mathbf{1})\boldsymbol{z}^{\mathbf{6}}$ ger tuple (x,y,z) satisfying the above equation are obtained. Various interesting properties among the values of x,y,z are presented.

**Notations**

$t_{m,n}$ **:** Polygonal number of rank n with size m

$CP_n^m$ : Centered Pyramidal number of rank n with size m.

$F_{4,n,7}$: Four dimensional heptagonal figurate number of rank n

$ky_n$ : keynea number of rank n.

$j_n$ : Jacobsthal lucas number of rank n

$J_n$ : Jacobsthal number of rank n

## MATERIALS AND METHODS

The non-homogeneous sextic equation with three unknowns to be solved for its distinct non-zero integral solutions is

$$(4k-1)(x^2+y^2)-(4k-2)xy=4(4k-1)z^6 \quad (1)$$

Introduction of the linear transformations

$$x=u+v, \quad y=u-v \quad (2)$$

In (1) leads to

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$$ku^2 + (3k-1)v^2 = (4k-1)z^6 \quad (3)$$

Different methods of obtaining the patterns of integer solutions to (1) are illustrated below:

**Pattern: 1**

Let

$$z = ka^2 + (3k-1)b^2 \quad (4)$$

Write $(4k-1)$ as

$$(4k-1) = \left(\sqrt{k} + i\sqrt{3k-1}\right)\left(\sqrt{k} - i\sqrt{3k-1}\right) \quad (5)$$

Using (4), (5) in (3) and applying the method of factorization, define

$$\left(\sqrt{k}u + iv\sqrt{3k-1}\right) = \left(\sqrt{k} + i\sqrt{3k-1}\right)\left(\alpha + i\beta\sqrt{3k-1}\sqrt{k}\right) \quad (6)$$

where $\left(\alpha + i\beta\sqrt{3k-1}\sqrt{k}\right) = \left(\sqrt{k}a + ib\sqrt{3k-1}\right)^6$

from which we have

$$\left.\begin{array}{l} \alpha = k^3a^6 - 15k^2(3k-1)a^4b^2 + 15k(3k-1)^2a^2b^4 - (3k-1)^3b^6 \\ \beta = b\,k^2a^5 - 20k(3k-10a^3b^3 + 6(3k-1)^2ab^5 \end{array}\right\} \quad (7)$$

Equating real and imaginary parts in (6), we have

$$\left.\begin{array}{l} u = \alpha - (3k-1)\beta \\ v = \alpha + \beta k \end{array}\right\} \quad (8)$$

Using (8) and (2), the values of x and y are given by

$$\left.\begin{array}{l} x(a,b) = 2\alpha - (2k-1)\beta \\ y(a,b) = -(4k-1)\beta \end{array}\right\} \quad (9)$$

Thus (4) and (9) represent the non-zero integer solutions to (1)

**Properties:**

(i) $(4k-1)x(1,n) - (2k-1)y(1,n) = 2(4k-1)\{k^3 - (3k-1)t_{4,n}[15k^2 + 15k(3k-1)t_{4,n} - (3k-1)^2\left(6F_{4,n,7} - 2CP_n^9 - 2CP_n^3 - CP_n^6 - 2t_{4,n}\right)]\}$

(ii) $y(1,n) = -(4k-1)[6k^2\left(Gn_n + t_{6,n} - 2t_{4,n} + 1\right) - 2(3k-1)CP_n^6\left[3(3k-1)t_{4,n} - 10k\right]]$

(iii) $z(2^n, 2^n) + (4k-1) = (4k-1)j_{2n}$

**Pattern:2**

Let
$$\left.\begin{array}{l} u = X + (3k-1)T \\ v = X - kT \end{array}\right\} \quad (10)$$

Substituting (10) in (3), we have

$$X^2 + k(3k-1)T^2 = z^6 \quad (11)$$

(11) can be written as

$$(z^3)^2 - X^2 = k(3k-1)T^2 \quad (12)$$

which is equivalent to the system of equations

$$\left.\begin{array}{l} z^3 + X = (3k-1)T \\ z^3 - X = kT \end{array}\right\} \quad (13)$$

Solving (13), we get

$z^3 = \frac{(4k-1)T}{2}$ and $X = \frac{(2k-1)T}{2}$

For Z and X to be integers, we choose T as

$$T = 2s^2(4k-1)^2 \quad (14)$$

Therefore

$$\left.\begin{aligned} z &= s(4k-1) \\ X &= s^3(2k-1)(4k-1)^2 \end{aligned}\right\} \quad (15)$$

Using (14), (15) in (10) we have

$$\left.\begin{aligned} u &= s^3(4k-1)^2(8k-3) \\ v &= -s^3(4k-1)^2 \end{aligned}\right\} \quad (16)$$

Using (2) and (16), the value of x and y satisfy (1) are given by

$$x = 4s^3(2k-1)(4k-1)^2$$
$$y = 2s^3(4k-1)^2$$
$$z = s(4k-1)$$

**Properties**

(i) $(4k-1)x = 2(2k-1)y$
(ii) $y = 2z^3$
(iii) $(4k-1)x = 4(2k-1)z^3$
(iv) $(4k-1)(2y-x) = 8kz^3$
(v) $4(2k-1)z^3 = (4k-1)x$
(vi) $32(2k-1)^3yz^6$ is a cubic number

**Pattern: 3**

(12) can be written as the system of double equations as

$$\left.\begin{aligned} z^3 + X &= k(3k-1)T \\ z^3 - X &= T \end{aligned}\right\} \quad (17)$$

Solving (17), we have

$z^3 = \frac{T(3k^2-k+1)}{2}$ and $X = \frac{T(3k^2-k-1)}{2}$

For Z and X to be integers, we choose T as

$$T = 2r^3(3k^2-k+1)^2 \quad (18)$$

Therefore

$$\left.\begin{aligned} z &= r(3k^2-k+1) \\ X &= r^3(3k^2-k+1)^2(3k^2-k-1) \end{aligned}\right\} \quad (19)$$

From (18), (19) and (10) we have

$$\left.\begin{aligned} u &= r^3(3k^2-k+1)^2(3k^2+5k-3) \\ v &= r^3(3k^2-k+1)^2(3k^2-3k-1) \end{aligned}\right\} \quad (20)$$

Substituting (20) in (2) we get the non-zero distinct integral solutions to (1) are

$$x = 2r^3(3k^2-k+1)^2(3k^2+k-2)$$

$$y = r^3(3k^2-k+1)^2(8k-2)$$

$z = r(3k^2 - k + 1)$

**Properties**

(i) $(4k-1)x = (3k^2 + k - 2)y$
(ii) $4(4k-1)^3 xz^6 = (3k^2 - k + 1)^2 (3k^2 + k - 2)y^3$
(iii) $(3k^2 - k + 1)y = (8k - 2)z^3$
(iv) $(3k^2 - k + 1)(x - 2z^3) = 2(2k - 3)z^3$

**Pattern: 4**

Instead of (10) we can write

$$\left.\begin{aligned} u &= x - (3k-1)T \\ v &= X + kT \end{aligned}\right\} \quad (21)$$

Using (21) in (3) and preceding as in pattern.3 the corresponding integer solutions to (1) are as follows

$x = r^3 6k(k-1)(3k^2 - k + 1)^2$

$y = -2r^3(3k^2 - k + 1)^2(4k + 1)$

$z = r(3k^2 - k + 1)$

**Properties**

(i) $(3k^2 - k + 1)x = 6k(k-1)z^3$
(ii) $(3k^2 - k + 1)y = -2(4k+1)z^3$
(iii) $(4k+1)x + 3k(k-1)y = 0$

**Pattern: 5**

(11) can be written as

$$(z^3)^2 = X^2 + k(3k-1)T^2 \quad (22)$$

where $k(3k-1) > 0$ , Assum $k(3k-1)$ is square free

Then, (22) is satisfied by

$$\left.\begin{aligned} T &= 2mn \\ X &= k(3k-1)m^2 - n^2 \end{aligned}\right\} \quad (23)$$

$$z^3 = k(3k-1)m^2 + n^2 \quad (24)$$

To find z, let $z = k(3k-1)A^2 + B^2$ (25)

Substituting (25) in (24) and employing the method of factorization, define

$(\sqrt{k(3k-1)}A + iB)^3 = [\sqrt{k(3k-1)}m + in]$

Equating real and imaginary parts, we have

$$\left.\begin{aligned} m &= A^3 k(3k-1) - 3AB^2 \\ n &= 3A^2 Bk(3k-1) - B^3 \end{aligned}\right\} \quad (26)$$

Using (23),(24),(26) and (10) we have

$$\left.\begin{aligned} u &= k(3k-1)m^2 - n^2 + 2(3k-1)mn \\ v &= k(3k-1)m^2 - n^2 - 2kmn \end{aligned}\right\} \quad (27)$$

Substituting (27) in (2), the values of x, y are given by

$$\left.\begin{array}{r} x = 2k(3k-1)m^2 - 2n^2 + 2(2k-1)mn \\ y = 2mn(4k-1) \end{array}\right\} \quad (28)$$

Thus, (25) and (28) represent the non-zero distinct integral solutions to (1)

**Pattern: 6**

Assume $k(3k-1)$ is a perfect square, say $M^2$ (29)

Then (11) becomes,

$$(z^3)^2 = X^2 + (MT)^2$$

which is satisfied by

$$MT = a^2 - b^2, \quad X = 2ab, \quad z^3 = a^2 + b^2 \quad (30)$$

Where $a > b > 0$

Replacing a by MA and b by MB in (30) we have

$$\left.\begin{array}{r} X = 2M^2AB \\ T = M^2(A^2 - B^2) \end{array}\right\} \quad (31)$$

$$z^3 = M^2(A^2 + B^2) \quad (32)$$

To find z:

Let

$$\left.\begin{array}{r} A = M^2 p_1(p_1^2 + q_1^2) \\ B = M^2 q_1(p_1^2 + q_1^2 \end{array}\right\} \quad (33)$$

Substituting (33) in (32) and employing the method of factorization, we have

$$z = M^2(p_1^2 + q_1^2) \quad (34)$$

Using (33) in (31), we get

$$\left.\begin{array}{r} X = 2M^6 p_1 q_1 (p_1^2 + q_1^2)^2 \\ T = M^6(p_1^2 + q_1^2)^2(p_1^2 - q_1^2) \end{array}\right\} \quad (35)$$

Using (35) in (21) and substituting in (2) we get the non-zero distinct integral solutions to (1) are found to be

$$x = M^6(p_1^2 + q_1^2)^2[4p_1q_1 + (p_1^2 - q_1^2)(2K-1)]$$

$$Y = M^6(p_1^2 + q_1^2)^2[(p_1^2 - q_1^2)(4K-1)]$$

$$Z = M^2(p_1^2 - q_1^2)$$

**Conclusion**

In this paper, we have made an attempt to determine different patterns of non-zero distinct integer solutions to the non-homogeneous sextic equation with three unknowns. As the sextic equations are rich in variety, one may search for other forms of sextic equation with variables greater than or equal to three and obtain their corresponding properties.

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